Homework 8, Math 250-07
due Wednesday, October 31, 2012

This home covers material from Sections 6.1 and 6.2. Refer there for more practice and help. You should practice the problems at the ends of these sections before starting this homework. Update: Due to class cancellation Monday, I have removed the last problem from this assignment.

1. Indefinite integrals. Classify the behavior of the following indefinite integrals. For integrals that do not converge, determine if the integral diverges to ∞, $-\infty$, or if there is a set of values among which the integral is indeterminate.
   Answer:*Note, you did not have to evaluate these, only describe their behavior*

   (a)
   $$\int_0^\infty xe^{-x}dx$$
   Answer:*converges because x does not grow as fast as e^x. The value can be calculated as $\mathcal{L}[x](s=1)$*

   (b)
   $$\int_0^\infty -x^{10}e^{-x}dx$$
   Answer:*also converges, and can be evaluated with the same Laplace-transform trick*

   (c)
   $$\int_0^\infty \cos(x+3)dx$$
   Answer:*bounded, but does not converge to any specific value*

   (d)
   $$\int_0^\infty x\sin(x^2)dx$$
   Answer:*by a change of variable, changes to the integral of $\sin(x)$, it's basically the same as the previous problem-- bounded but does not converge to any particular value.*

   (e)
   $$\int_0^\infty \frac{2+x}{3+x^2}dx$$
   Answer:*diverges, but diverges slowly, like $\ln x$.*

   (f)
   $$\int_0^\infty \sin(3x)e^x dx$$
   Answer:*has diverging oscillations to both positive and negative infinity*

2. Calculate the following Laplace transforms, and provide a range of values for s for which the transform exists. Use tables and rules where-ever possible to simplify your work, but remember to cite your methods.

   (a) $\mathcal{L}\{4e^{13t}\}$
   Answer: $4/(s-13)$

   (b) $\mathcal{L}\{\sinh(5t)\}$
   Answer: $\dfrac{1}{2(s-5)} - \dfrac{1}{2(s+5)}$

   (c) $\mathcal{L}\{-15t^4\}$
   Answer:$-15(4!)/s^5$

(d) $\mathcal{L}\{e^{6t}\sin(8t)\}$
Answer:$8/[(s-6)^2+64]$

(e) $\mathcal{L}\{te^t + t\cos(t)\}$
Answer: *To do these, we have to fall back on the definition of Laplace transforms, and integrate by parts.*

$$\frac{1}{(s-1)^2} + \frac{s^2-1}{(s^2+1)^2}$$

3. Show that

$$\frac{\partial}{\partial s}\mathcal{L}\{f(t)\} = -\mathcal{L}\{tf(t)\}.$$

Use this theorem to calculate $\mathcal{L}\{t\sin(4t)\}$.
Answer:

$$\begin{aligned}
\frac{\partial}{\partial s}\mathcal{L}\{f(t)\} &= \frac{\partial}{\partial s}\int_0^\infty e^{-st}f(t)dt \\
&= \int_0^\infty f(t)\frac{\partial}{\partial s}e^{-st}dt \\
&= \int_0^\infty f(t)(-t)e^{-st}dt \\
&= \mathcal{L}\{(-t)f(t)\} = -\mathcal{L}\{tf(t)\}
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{L}\{t\sin(4t)\} &= -\partial/\partial s\mathcal{L}(\sin(4t)) \\
&= -\partial/\partial s\left(\frac{4}{s^2+16}\right) \\
&= -\partial/\partial s\left(\frac{4}{s^2+16}\right) \\
&= \frac{8s}{(s^2+16)^2}
\end{aligned}$$

4. Inverting Laplace transforms requires practice. Calculate the following inverses.

(a)

$$\mathcal{L}^{-1}\left\{\frac{4}{s^3} - \frac{18}{s+3} + 2\right\}$$

Answer:

$$2t^2 - 18e^{-3t} + 2\delta(t)$$

(b)

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-4s-21}\right\}$$

Answer:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-7)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{10(s-7)} - \frac{1}{10(s+3)}\right\} = \frac{e^{7t}-e^{-3t}}{10}$$

(c) Use completing-the square to calculate the following inverse.

$$\mathcal{L}^{-1}\left\{\frac{-6}{s^2-8s+25}\right\}.$$

Answer:

$$\mathcal{L}^{-1}\left\{\frac{-6}{s^2-8s+25}\right\} = -2e^{4t}\sin(3t)$$

(d)

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3-1}\right\}$$

Answer: *Solving the cubic, we the roots of the denominator are* 1 *and* $(-1 \pm \sqrt{3})/2$. *Using partial fractions and completing-the-square, we can show*

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\left[\frac{1}{s-1} + \frac{3/2+s}{(s+1/2)^2+3/4}\right]\right\}$$

Inverting, we find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3-1}\right\} = -\frac{1}{3}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right) + \cos\left(\frac{\sqrt{3}}{2}t\right)\right)e^{-\frac{1}{2}t} + \frac{1}{3}e^t$$