Homework 7, Math 250-07
due Wednesday, October 24, 2012

This home covers material from Sections 5.2 and 5.3. Refer there for more practice and help. I urge you to do as many problems in the textbook as possible in addition to this assignment.

1. The Fibonnaci numbers are given by the linear recurrence equations

$$a_0 = 0, \quad a_1 = 1, \quad a_{n+2} = a_{n+1} + a_n.$$

Calculate the first 8 Fibonnaci numbers. Answer: *Fibonnaci's sequence is one of the most famous and mysterious in mathematics, because of it's frequent appearance in nature.*

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

2. Section 5.2 of the textbook, problems 1,2,3 and 4. Answer:*(if you got the closed-form series, great, but not so important here)*

5.2.1 $y'' - y = 0$, $x_0 = 0$.
Answer:

$$a_n = \frac{a_{n-2}}{n(n-1)}$$
$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$
$$y_1(x) = \frac{1}{720}x^6 + \frac{1}{24}x^4 + \frac{1}{2}x^2 + 1 + O(x^8)$$
$$y_2(x) = \frac{1}{5040}x^7 + \frac{1}{120}x^5 + \frac{1}{6}x^3 + x + O(x^9)$$
$$y_1(0) = 1,\ y_1'(0) = 0,\ y_2(0) = 0,\ y_2'(0) = 1$$
$$W(0) = y_1(0)y_2'(0) - y_2(0)y_1'(0) = 1 \neq 0$$
$$y(x) = C_1 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + C_2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = C_1 \cosh(x) + C_2 \sinh(x).$$

5.2.2 $y'' - xy' - y = 0,\ x_0 = 0$ Answer:

$$a_n = \frac{a_{n-2}}{n}$$
$$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + O(x^8)$$
$$y_2(x) = x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7 + O(x^9)$$
$$W(0) = y_1(0)y_2'(0) - y_2(0)y_1'(0) = (1)(1) - (0)(0) = 1$$

5.2.3 $y'' - xy' - y = 0$, $x_0 = 1$.
Answer:

$$a_n = \frac{a_{n-1} + a_{n-2}}{n}$$
$$y_1(x) = \frac{1}{6}(x-1)^4 + \frac{1}{6}(x-1)^3 + \frac{1}{2}(x-1)^2 + 1 + O((x-1)^5),$$
$$y_2(x) = x - 1 + \frac{1}{4}(x-1)^4 + \frac{1}{2}(x-1)^3 + \frac{1}{2}(x-1)^2 + O((x-1)^5)$$
$$W(1) = y_1(1)y_2'(1) - y_2(1)y_1'(1) = (1)(1) - (0)(0) = 1$$

5.2.4 $y'' + k^2x^2y = 0$, $x_0 = 0$.
Answer:

$$a_2 = 0, \quad a_3 = 0, \quad a_n = -\frac{k^2 a_{n-4}}{n(n-1)} \text{ when } n > 3.$$
$$y_1(x) = -\frac{1}{88704}k^6x^{12} + \frac{1}{672}k^4x^8 - \frac{1}{12}k^2x^4 + 1$$
$$y_2(x) = -\frac{1}{224640}k^6x^{13} + \frac{1}{1440}k^4x^9 - \frac{1}{20}k^2x^5 + x$$
$$W(0) = y_1(0)y_2'(0) - y_2(0)y_1'(0) = (1)(1) - (0)(0) = 1$$

3. Section 5.3 of the textbook problems 1,2,5,6,7.

5.3.1 $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$. Answer:

$$\mathrm{y}(x) = 1$$
$$\frac{\partial}{\partial x}\mathrm{y}(x) = 0$$
$$\frac{\partial^2}{\partial^2 x}\mathrm{y}(x) = -1$$
$$\frac{\partial^3}{\partial^3 x}\mathrm{y}(x) = 0$$
$$\frac{\partial^4}{\partial^4 x}\mathrm{y}(x) = 3$$

5.3.2 $y'' + \sin(x)y' + \cos(x)y = 0$, $y(0) = 0$, $y'(0) = 1$. Answer:

$$\mathrm{y}(x) = 0$$
$$\frac{\partial}{\partial x}\mathrm{y}(x) = 1$$
$$\frac{\partial^2}{\partial^2 x}\mathrm{y}(x) = 0$$
$$\frac{\partial^3}{\partial^3 x}\mathrm{y}(x) = -2$$
$$\frac{\partial^4}{\partial^4 x}\mathrm{y}(x) = 0$$

5.3.5 $y'' + 4y' + 6xy = 0$, with $x_0 = 0$ or $x_0 = 4$.

Answer: *There are no singular points in this equation, real or complex, so the radius of convergence is infinite for both $x_0 = 0$ and $x_0 = 4$.*

5.3.6 $(x^2 - 2x - 3)y'' + xy' + 4y = 0$ with $x_0 \in \{4, -4, 0\}$.

Answer: *The equation has singular points at $x = 3$ and $x = -1$. If $x_0 = 4$, then the radius of convergence $\rho = 1$ and our series will converge atleast for $x \in (3, 5)$. If $x_0 = -4$, then our radius of convergence $\rho = 3$, and our series will converge atleast for $x \in (-7, -1)$. If $x_0 = 0$, $\rho = 1$, and our series converges for $x \in (-1, 1)$.*

5.3.7 $(1 - x^3)y'' + 4xy' + y = 0$ with $x_0 \in \{0, 2\}$.

Answer: *There are 3 singular points for this equation,*

$$x \in \left\{-1,\ \frac{1}{2} - \imath\frac{1}{2}\sqrt{3},\ \frac{1}{2} + \imath\frac{1}{2}\sqrt{3}\right\}$$

If $x_0 = 0$, each is equidistant, so $\rho = 1$, and we can expect convergence for $x \in (-1, 1)$. If $x_0 = 2$, then the nearest singularities are in the complex plane.

$$\rho = \sqrt{\left(2 - \frac{1}{2}\right)^2 + \left(-\frac{1}{2}\sqrt{3}\right)^2} = \sqrt{3}$$

so our series will converge at least for $x \in (2 - \sqrt{3}, 2 + \sqrt{3})$.