Homework 6, Math 250-07
due Wednesday, October 17, 2012

This home covers material from Sections 3.7 and 5.1. Refer there for more practice and help. I urge you to do as many problems in the textbook as possible in addition to this assignment. Note: 1 typo in the summations below have been fixed. # 5 should start at $k = 0$. # 6 correctly starts at k=1.

1. A spring with relaxed length 3m hangs from the ceiling of a hanger on a planet where gravity's acceleration is $10m/s^2$.

   (a) A weight of mass 3 kg is hung from the spring. The spring stretches so that it's new rest length is 97/29 meters. What is the spring constant?
   Answer: 87

   (b) On this planet, we find that a wind velocity of 2 m/s is needed to suspend the weight against gravity. What is the coefficient of drag?
   Answer: *By balance-of-force, we should have* $\gamma\dot{u} = mg$ *in this situation.* $2\gamma = 30$, $\gamma = 15$

   (c) Show that this spring system is under-damped and determine the quasi-period of oscillations?
   Answer: *For the system to be under-damped, we must have* $\gamma^2 < 4km$, *or* $15^2 < 4 \times 87 \times 3$. *The quasiperiod* $T_q = 2\pi\sqrt{\frac{4m^2}{4mk-\gamma^2}} = \frac{4\pi}{\sqrt{91}}$

   (d) Formulate the initial value problem for the displacement of the weight from rest position when the weight starts with initial displacement of 3 m from rest and with an initial velocity of -16 m/s.
   Answer:
   $$3u'' + 15u' + 87u = 0, \quad u(0) = 3, \ u'(0) = -16.$$
   or
   $$u'' + 5u' + 29u = 0, \quad u(0) = 3, \ u'(0) = -16.$$

   (e) Find the general solution for the displacement. Answer:
   $$u(t) = e^{-5t/2}\left(C_1 \sin(\sqrt{91}t/2) + C_2 \cos(\sqrt{91}t/2)\right)$$

   (f) Find the specific solution matching the initial conditions.
   Answer:
   $$u(t) = e^{-5t/2}\left(\frac{-17}{\sqrt{91}} \sin(\sqrt{91}t/2) + 3\cos(\sqrt{91}t/2)\right)$$

   (g) Find a specific solution for the length of the spring for all time t. Answer:
   $$u(t) = \frac{97}{29} + e^{-5t/2}\left(\frac{-17}{\sqrt{91}} \sin(\sqrt{91}t/2) + 3\cos(\sqrt{91}t/2)\right)$$

2. Consider a harmonic oscillator, where the spring has spring-coefficient $k = 192$ N/m and the suspended mass has weight $m = 3$ kg.

   (a) What's the period of oscillations of this system when there is no damping?
   Answer: $T = 2\pi\sqrt{m/k} = \frac{1}{4}\pi$

   (b) How strong must the viscuous damping of the system be for the dynamics to exhibit critical damping?
   Answer: *The critical value for the drag coefficient* $\gamma_c = 2\sqrt{mk} = 48$.

   (c) Find the general solution for the motion of the mass if the drag coefficient is twice the critical value?
   Answer:
   $$u(t) = C_1 e^{t\left(-16+8\sqrt{3}\right)} + C_2 e^{t\left(-16-8\sqrt{3}\right)}$$

3. Find a_2 when $\sum_{n=0}^{\infty} a_n t^n := y(t) = (-4t-4)\,e^{-5t} + (-3t+5)\cos(2t)$.

Answer:

$$
\begin{aligned}
-4te^{-5t} &= -4t + 20t^2 - 50t^3 + \mathcal{O}\left(t^4\right), \\
-4e^{-5t} &= -4 + 20t - 50t^2 + \frac{250}{3}t^3 + \mathcal{O}\left(t^4\right), \\
-3t\cos(2t) &= -3t + 6t^3 + \mathcal{O}\left(t^4\right), \\
5\cos(2t) &= 5 - 10t^2 + \mathcal{O}\left(t^4\right) \\
a_2 &= 20 - 50 - 10 = -40.
\end{aligned}
$$

4. Find a_4 when $\sum_{n=0}^{\infty} a_n t^n := y(t) = (-5t-3)\sin(3t) + (3t+5)\,e^{-4t}$.

Answer:

$$
\begin{aligned}
-5t\sin(3t) &= -15t^2 + \frac{45}{2}t^4 + \mathcal{O}\left(t^5\right) \\
-3\sin(3t) &= -9t + \frac{27}{2}t^3 + \mathcal{O}\left(t^5\right) \\
3te^{-4t} &= 3t - 12t^2 + 24t^3 - 32t^4 + \mathcal{O}\left(t^5\right) \\
5e^{-4t} &= 5 - 20t + 40t^2 - \frac{160}{3}t^3 + \frac{160}{3}t^4 + \mathcal{O}\left(t^5\right) \\
a_4 &= \frac{263}{6}
\end{aligned}
$$

5. Show that the infinite series

$$y(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}$$

is equal to it's own derivative and solves

$$\frac{dy}{dt} = y.$$

Answer: *By direct calculation,*

$$
\begin{aligned}
y'(t) &= \sum_{k=0}^{\infty} k\frac{t^{k-1}}{k!} \\
y'(t) &= \sum_{k=1}^{\infty} k\frac{t^{k-1}}{k!} \\
y'(t) &= \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!}
\end{aligned}
$$

If we let $m+1=k$,

$$
\begin{aligned}
y'(t) &= \sum_{m+1=1}^{\infty} \frac{t^{(m+1)-1}}{((m+1)-1)!} \\
y'(t) &= \sum_{m=0}^{\infty} \frac{t^m}{m!} = y(t).
\end{aligned}
$$

6. Given that

$$y(t) = \sum_{k=1}^{\infty} \frac{(-2)^k t^k}{k!},$$

calculate

$$\frac{d^2y}{dt^2} - y$$

.

Answer:

$$\frac{d^2y}{dt^2} - y = 1 + 3e^{-2t}$$

.

7. Use the Maclaurin series for e^x, $\sin x$ and $\cos x$ to show Euler's formula

$$e^{it} = \cos t + i\sin(t).$$

Answer: *One way (but there are other ways just as good and better) is*

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}$$

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^{(}2n)}{(2n)!} + \frac{(it)^{(}2n+1)}{(2n+1)!}$$

$$e^{it} = \sum_{n=0}^{\infty} \frac{(i)^{(}2n)(t)^{(}2n)}{(2n)!} + \frac{i(i)^{(}2n)(t)^{(}2n+1)}{(2n+1)!}$$

$$e^{it} = \sum_{n=0}^{\infty} \frac{(-1)^{(}n)(t)^{(}2n)}{(2n)!} + \frac{i(-1)^{(}n)(t)^{(}2n+1)}{(2n+1)!}$$

$$e^{it} = \sum_{n=0}^{\infty} \frac{(-1)^{(}n)(t)^{(}2n)}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^{(}n)(t)^{(}2n+1)}{(2n+1)!}$$

$$e^{it} = \cos(t) + i\sin(t)$$