1. A spring with relaxed length 3m hangs from the ceiling of a hanger on a planet where gravity’s acceleration is 10 m/s^2.

   (a) A weight of mass 3 kg is hung from the spring. The spring stretches so that it’s new rest length is 97/29 meters. What is the spring constant?
   Answer: 87

   (b) On this planet, we find that a wind velocity of 2 m/s is needed to suspend the weight against gravity. What is the coefficient of drag?
   Answer: By balance-of-force, we should have γ\ddot{u} = mg in this situation.

   \[ \gamma = \frac{2 \times 10}{2} = 15 \]

   (c) Show that this spring system is under-damped and determine the quasi-period of oscillations?
   Answer: For the system to be under-damped, we must have \( \gamma^2 < 4mk \), or \( 15^2 < 4 \times 87 \times 3 \). The quasi-period \( T_q = \frac{2\pi}{\sqrt{\frac{4m^2}{4mk} - \gamma^2}} = \frac{4\pi}{\sqrt{91}} \)

   (d) Formulate the initial value problem for the displacement of the weight from rest position when the weight starts with initial displacement of 3 m from rest and with an initial velocity of -16 m/s.
   Answer: 
   \[ 3u'' + 15u' + 87u = 0, \quad u(0) = 3, \quad u'(0) = -16. \]
   or
   \[ u'' + 5u' + 29u = 0, \quad u(0) = 3, \quad u'(0) = -16. \]

   (e) Find the general solution for the displacement. Answer:
   \[ u(t) = e^{-5t/2} \left( C_1 \sin(\sqrt{91}t/2) + C_2 \cos(\sqrt{91}t/2) \right) \]

   (f) Find the specific solution matching the initial conditions.
   Answer:
   \[ u(t) = e^{-5t/2} \left( -\frac{17}{\sqrt{91}} \sin(\sqrt{91}t/2) + 3 \cos(\sqrt{91}t/2) \right) \]

   (g) Find a specific solution for the length of the spring for all time \( t \). Answer:
   \[ u(t) = \frac{97}{29} + e^{-5t/2} \left( -\frac{17}{\sqrt{91}} \sin(\sqrt{91}t/2) + 3 \cos(\sqrt{91}t/2) \right) \]

2. Consider a harmonic oscillator, where the spring has spring-coefficient \( k = 192 \) N/m and the suspended mass has weight \( m = 3 \) kg.

   (a) What’s the period of oscillations of this system when there is no damping?
   Answer: \( T = 2\pi \sqrt{m/k} = \frac{1}{3} \pi \)

   (b) How strong must the viscous damping of the system be for the dynamics to exhibit critical damping?
   Answer: The critical value for the drag coefficient \( \gamma_c = 2\sqrt{mk} = 48 \).

   (c) Find the general solution for the motion of the mass if the drag coefficient is twice the critical value?
   Answer:
   \[ u(t) = C_1 e^{t(-16+8\sqrt{3})} + C_2 e^{t(-16-8\sqrt{3})} \]
3. Find \( a_2 \) when \( \sum_{n=0}^{\infty} a_n t^n := y(t) = (-4t - 4) e^{-5t} + (-3t + 5) \cos (2t) \).

Answer:
\[
\begin{align*}
-4te^{-5t} &= -4t + 20t^2 - 50t^3 + O(t^4), \\
-4e^{-5t} &= -4 + 20t - 50t^2 + \frac{250}{3}t^3 + O(t^4), \\
-3t \cos (2t) &= -3t + 6t^3 + O(t^4), \\
5 \cos (2t) &= 5 - 10t^2 + O(t^4) \\
a_2 &= 20 - 50 - 10 = -40.
\end{align*}
\]

4. Find \( a_4 \) when \( \sum_{n=0}^{\infty} a_n t^n := y(t) = (-5t - 3) \sin (3t) + (3t + 5) e^{-4t} \).

Answer:
\[
\begin{align*}
-5t \sin (3t) &= -15t^2 + \frac{45}{2} t^3 + O(t^5) \\
-3 \sin (3t) &= -9t + \frac{27}{2} t^3 + O(t^5) \\
3te^{-4t} &= 3t - 12t^2 + 24t^3 - 32t^4 + O(t^5) \\
5e^{-4t} &= 5 - 20t + 40t^2 - \frac{160}{3} t^3 + \frac{160}{3} t^4 + O(t^5) \\
a_4 &= \frac{263}{6}.
\end{align*}
\]

5. Show that the infinite series
\[
y(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}
\]
is equal to it’s own derivative and solves
\[
\frac{dy}{dt} = y.
\]

Answer: By direct calculation,
\[
\begin{align*}
y′(t) &= \sum_{k=0}^{\infty} \frac{t^{k-1}}{k!} \\
y′(t) &= \sum_{k=1}^{\infty} \frac{t^{k-1}}{k!} \\
y′(t) &= \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!}
\end{align*}
\]
If we let \( m + 1 = k \),
\[
\begin{align*}
y′(t) &= \sum_{m+1=1}^{\infty} \frac{t^{(m+1)-1}}{((m+1) - 1)!} \\
y′(t) &= \sum_{m=0}^{\infty} \frac{t^m}{m!} = y(t).
\end{align*}
\]

6. Given that
\[
y(t) = \sum_{k=1}^{\infty} \frac{(-2)^k t^k}{k!},
\]
calculate
\[
\frac{d^2 y}{dt^2} = y.
\]

Answer:
\[
\frac{d^2 y}{dt^2} = y = 1 + 3e^{-2t}.
\]
7. Use the Maclaurin series for $e^x$, $\sin x$ and $\cos x$ to show Euler’s formula

$$e^{it} = \cos t + i \sin(t).$$

Answer: One way (but there are other ways just as good and better) is

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}$$

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^{2n}}{(2n)!} + \frac{(it)^{2n+1}}{(2n+1)!}$$

$$e^{it} = \sum_{n=0}^{\infty} \frac{(i)^n(2n)(t)^{(2n)}}{(2n)!} + \frac{i(i)^n(2n)(t)^{(2n+1)}}{(2n+1)!}$$

$$e^{it} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)(t)^{(2n)}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n(2n)(t)^{(2n+1)}}{(2n+1)!}$$

$$e^{it} = \cos(t) + i \sin(t)$$