Homework 5, Math 250-07
due Wednesday, October 10, 2012

This home covers material from Sections 3.5, 4.1, and 4.2. Refer there for more practice and help. I urge you to do as many problems in the textbook as possible in addition to this assignment.

1. Consider the differential equation

$$-63\,\mathrm{y}(x) - 16\frac{\partial}{\partial x}\mathrm{y}(x) - \frac{\partial^2}{\partial^2 x}\mathrm{y}(x) = -252x^2 + 187x + 198, \quad \mathrm{y}(0) + 17 = 0, \quad \mathrm{y'}(0) - 126 = 0.$$

   (a) Find the general solution of the homogenized equation.
   Answer:

$$y_h(x) = C_1 e^{-9x} + C_2 e^{-7x}$$

   (b) Find a particular solution using the method of undetermined coefficients.
   Answer:

$$-63\,\mathrm{y}(x) - 16\frac{\partial}{\partial x}\mathrm{y}(x) - \frac{\partial^2}{\partial^2 x}\mathrm{y}(x) = -252x^2 + 187x + 198$$

$$63C_0 + 63C_1x - 63C_2x^2 + 252x^2 - 187x - 16\frac{\partial}{\partial x}\left(-C_0 - C_1x + C_2x^2\right) - \frac{\partial^2}{\partial^2 x}\left(-C_0 - C_1x + C_2x^2\right) - 198$$

$$63C_0 + 63C_1x + 16C_1 - 63C_2x^2 - 32C_2x - 2C_2 + 252x^2 - 187x - 198$$

$$63C_0 + 16C_1 - 2C_2 + x^2\left(-63C_2 + 252\right) + x\left(63C_1 - 32C_2 - 187\right) - 198$$

$$\left[63C_0 + 16C_1 - 2C_2 - 198, \quad 63C_1 - 32C_2 - 187, \quad -63C_2 + 252\right]$$

$$\left\{C_0 : 2, \quad C_1 : 5, \quad C_2 : 4\right\}$$

$$y_p(x) = 4x^2 - 5x - 2$$

   (c) Find the specific solution to the given problem.
   Answer:

$$\mathrm{y}(x) = C_0e^{-9x} + C_1e^{-7x} + 4x^2 - 5x - 2$$

$$C_0 + C_1 + 15 = 0$$

$$-9C_0 - 7C_1 - 131 = 0$$

$$\left\{C_0 : -13, \quad C_1 : -2\right\}$$

$$\mathrm{y}(x) = 4x^2 - 5x - 2 - 2e^{-7x} - 13e^{-9x}$$

2. Consider the differential equation

$$-66\,\mathrm{y}(x) - 17\frac{\partial}{\partial x}\mathrm{y}(x) - \frac{\partial^2}{\partial^2 x}\mathrm{y}(x) = 756e^x, \quad \mathrm{y}(0) + 13 = 0, \quad \mathrm{y'}(0) - 40 = 0.$$

   (a) Find the general solution of the homogenized equation.
   Answer:

$$y_h(x) = C_1e^{-6x} + C_2e^{-11x}$$

   (b) Find a particular solution using the method of undetermined coefficients.
   Answer:

$$-66\,\mathrm{y}(x) - 17\frac{\partial}{\partial x}\mathrm{y}(x) - \frac{\partial^2}{\partial^2 x}\mathrm{y}(x) = 756e^x$$

$$-66Ce^x - 756e^x - 17\frac{\partial}{\partial x}\left(Ce^x\right) - \frac{\partial^2}{\partial^2 x}\left(Ce^x\right)$$

$$-84Ce^x - 756e^x$$

$$-84\left(C + 9\right)e^x$$

$$C = -9$$

$$\mathrm{y\_p}(x) = -9e^x$$

(c) Find the specific solution to the given problem.
Answer:

$$\mathrm{y}(x) = C_1 e^{-6x} + C_2 e^{-11x} - 9e^{x}$$
$$C_1 + C_2 + 4 = 0$$
$$-6C_1 - 11C_2 - 49 = 0$$
$$\{C_1 : 1, \quad C_2 : -5\}$$
$$\mathrm{y}(x) = -9e^{x} + e^{-6x} - 5e^{-11x}$$

3. Consider the differential equation

$$\frac{\partial}{\partial x}\mathrm{y}(x) + \frac{\partial^2}{\partial^2 x}\mathrm{y}(x) = 24\sin(4x) + 96\cos(4x), \quad \mathrm{y}(0) + 4 = 0, \quad \mathrm{y'}(0) + 4 = 0.$$

(a) Find the general solution of the homogenized equation.
Answer:

$$y_h(x) = C_1 1 + C_2 e^{-x}$$

(b) Find a particular solution using the method of undetermined coefficients.
Answer:

$$\frac{\partial}{\partial x}\mathrm{y}(x) + \frac{\partial^2}{\partial^2 x}\mathrm{y}(x) = 24\sin(4x) + 96\cos(4x)$$
$$-24\sin(4x) - 96\cos(4x) + \frac{\partial}{\partial x}(-C_1\sin(4x) + C_2\cos(4x)) + \frac{\partial^2}{\partial^2 x}(-C_1\sin(4x) + C_2\cos(4x))$$
$$-4(-4C_1\sin(4x) + C_1\cos(4x) + C_2\sin(4x) + 4C_2\cos(4x) + 6\sin(4x) + 24\cos(4x))$$
$$-4(-4C_1 + C_2 + 6)\sin(4x) - 4(C_1 + 4C_2 + 24)\cos(4x)$$
$$[16C_1 - 4C_2 - 24, \quad -4C_1 - 16C_2 - 96]$$
$$\{C_1 : 0, \quad C_2 : -6\}$$
$$y_p(x) = -6\cos(4x)$$

(c) Find the specific solution to the given problem.
Answer:

$$\mathrm{y}(x) = C_1 + C_2 e^{-x} - 6\cos(4x)$$
$$C_1 + C_2 - 2 = 0$$
$$-C_2 + 4 = 0$$
$$\{C_1 : -2, \quad C_2 : 4\}$$
$$\mathrm{y}(x) = -6\cos(4x) - 2 + 4e^{-x}$$

4. Consider the differential equation

$$-16\,\mathrm{y}(t) - 8\frac{\partial}{\partial t}\mathrm{y}(t) - \frac{\partial^2}{\partial^2 t}\mathrm{y}(t) = 8e^{-4t}, \quad \mathrm{y}(0) - 10 = 0, \quad \mathrm{y'}(0) + 53 = 0.$$

(a) Find the general solution of the homogenized equation.
Answer:

$$y_h(t) = C_1 e^{-4t} + C_2 t e^{-4t}$$

(b) Find a particular solution using the method of undetermined coefficients.

Answer:

$$-16\,\mathrm{y}(t) - 8\frac{\partial}{\partial t}\mathrm{y}(t) - \frac{\partial^2}{\partial^2 t}\mathrm{y}(t) = 8e^{-4t}$$
$$\mathrm{y}(t) = Ct^2e^{-4t}$$
$$-16Ct^2e^{-4t} - 8\frac{\partial}{\partial t}\left(Ct^2e^{-4t}\right) - \frac{\partial^2}{\partial^2 t}\left(Ct^2e^{-4t}\right) - 8e^{-4t}$$
$$-2Ce^{-4t} - 8e^{-4t}$$
$$-2(C+4)e^{-4t}$$
$$C = -4$$
$$\mathrm{y\_p}(t) = -4t^2e^{-4t}$$

(c) Find the specific solution to the given problem.
Answer:

$$\mathrm{y}(t) = C_1e^{-4t} + C_2te^{-4t} - 4t^2e^{-4t}$$
$$C_1 - 10 = 0$$
$$-4C_1 + C_2 + 53 = 0$$
$$\{C_1 : 10, \quad C_2 : -13\}$$
$$\mathrm{y}(t) = -4t^2e^{-4t} - 13te^{-4t} + 10e^{-4t}$$

5. Consider the differential equation

$$8\,\mathrm{y}(x) - 4\frac{\partial}{\partial x}\mathrm{y}(x) + \frac{\partial^2}{\partial^2 x}\mathrm{y}(x) = -8x + 60, \quad \mathrm{y}(0) - 1 = 0, \quad \mathrm{y'}(0) + 19 = 0.$$

(a) Find the general solution of the homogenized equation.
Answer:
$$y_h(x) = C_1e^{2x}\cos(2x) + C_2e^{2x}\sin(2x)$$

(b) Find a particular solution using the method of undetermined coefficients.
Answer:

$$8\,\mathrm{y}(x) - 4\frac{\partial}{\partial x}\mathrm{y}(x) + \frac{\partial^2}{\partial^2 x}\mathrm{y}(x) = -8x + 60$$
$$8C_0 - 8C_1x - 8C_2x^2 + 8x - 4\frac{\partial}{\partial x}\left(C_0 - C_1x - C_2x^2\right) + \frac{\partial^2}{\partial^2 x}\left(C_0 - C_1x - C_2x^2\right) - 60$$
$$-2\left(-4C_0 + 4C_1x - 2C_1 + 4C_2x^2 - 4C_2x + C_2 - 4x + 30\right)$$
$$8C_0 + 4C_1 - 8C_2x^2 - 2C_2 + x(-8C_1 + 8C_2 + 8) - 60$$
$$\left[8C_0 + 4C_1 - 2C_2 - 60, \quad -8C_1 + 8C_2 + 8, \quad -8C_2\right]$$
$$\{C_0 : 7, \quad C_1 : 1, \quad C_2 : 0\}$$
$$y_p(x) = -x + 7$$

(c) Find the specific solution to the given problem.
Answer:

$$\mathrm{y}(x) = C_0e^{2x}\cos(2x) + C_1e^{2x}\sin(2x) - x + 7$$
$$C_0 + 6 = 0$$
$$2C_0 + 2C_1 + 18 = 0$$
$$\{C_0 : -6, \quad C_1 : -3\}$$
$$\mathrm{y}(x) = -x - 3e^{2x}\sin(2x) - 6e^{2x}\cos(2x) + 7$$

6. Consider the set of functions
$$\{\cos(4t), \sin(4t), e^{3t}\}.$$

(a) Find a 3rd order linear constant coefficient homogeneous ordinary differential equation for which this is the fundamental solution set. Answer: *The roots of the characteristic equation are* $r \in \{4i, -4i, 3\}$. *So our characteristic equation is*

$$(r-4i)(r+4i)(r-3) = 0.$$

$$(r^2+16)(r-3) = 0.$$

$$r^3 - 3r^2 + 16r - 48 = 0.$$

$$y''' - 3y'' + 16y' - 48y = 0.$$

(b) If $y(0) = 0$, $y'(0) = 1$, $y''(0) = -1$, then the specific solution of this equation is what? Answer: $C_0 = 1/25, C_1 = 7/25, C_2 = -1/25$

7. Find the general solution of the 3rd-order equation

$$y''' + 2y = 0.$$

Answer: *Our characteristic equation is* $r^3 + 2 = 0$. *A sum of cubes* $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$. *Using this to find 2 complex roots, we determine*

$$y(t) = C_1 e^{\frac{1}{2}\sqrt[3]{2}t} \sin\left(\frac{1}{2}\sqrt[3]{2}\sqrt{3}t\right) + C_2 e^{\frac{1}{2}\sqrt[3]{2}t} \cos\left(\frac{1}{2}\sqrt[3]{2}\sqrt{3}t\right) + C_3 e^{-\sqrt[3]{2}t}$$

8. A set of functions solving an nth-order linear equation is only a fundamental solution set if there are n of them, and they are all "independent". The functions are independent if none of them can be expressed as the sums of the others. So $\{e^t, e^{2t}, e^{3t}\}$ might be a fundamental solution set of a 3rd order equation, but $\{e^t, e^{2t}, e^t + 2e^{2t}\}$ is not. A set of functions is independent if it's Wronskian is non-zero, but we can also test for independence directly sometimes. Explain why $\{\sin(t), \cos(t), \cos(t - \pi/4)\}$ can not be a fundamental solution set of a 3rd order equation. Answer: *Using the trig sum formulas, we can show*

$$\cos(t - \pi/4) = \cos(t)\cos(\pi/4) + \sin(t)\sin(\pi/4)$$

so the 3rd function is a (scaled) sum of the first two.