Homework 3, Math 250-07

due Friday, September 21, 2012

This home covers material from Sections 2.5, and 3.1. Refer there for more practice and help. I urge you to do as many problems in the textbook as possible in addition to this assignment.

1. Consider the differential equation

\[ \frac{\partial}{\partial t} x(t) = -x^2 - x + 6. \]

(a) Are there any places where Theorem 2.4.2 does not apply? Answer: This is a nonlinear autonomous equation. The right hand side is continuous everywhere, so Theorem 2.4.2 always applies.

(b) Find all stationary solutions. Answer: The set of stationary roots is \{2, -3\}.

(c) For each stationary solution, determine if it is stable, unstable, semistable. Answer: \(x(t) = -3\) is unstable. \(x(t) = 2\) is stable.

(d) For each possible initial condition, determine the asymptotic behavior of the corresponding solution. Answer: If \(x(0) \in (-\infty, -3)\), then \(x(t)\) diverges to \(-\infty\). If \(x(0) \in (-3, \infty)\), then \(x(t)\) converges to 2. If \(x(0) = -3\), then \(x(t) = -3\) always.

2. Consider the differential equation

\[ \frac{\partial}{\partial t} x(t) = -x^2 - 16x - 64. \]

(a) Are there any places where Theorem 2.4.2 does not apply? Answer: This is a nonlinear autonomous equation. The right hand side is continuous everywhere, so Theorem 2.4.2 always applies.

(b) Find all stationary solutions. Answer: The set of stationary roots is \{-8\}.

(c) For each stationary solution, determine if it is stable, unstable, semistable. Answer: \(x(t) = -8\) is semistable.

3. Consider the differential equation

\[ \frac{\partial}{\partial t} x(t) = e^{x^3 + 4x^2 - 17x - 60} - 1. \]

(a) Are there any places where Theorem 2.4.2 does not apply? Answer: This is a nonlinear autonomous equation. The right hand side is continuous everywhere, so Theorem 2.4.2 always applies.

(b) Find all stationary solutions. Answer: The stationary roots are \([-5, -3, 4]\).

(c) For each stationary solution, determine if it is stable, unstable, semistable. Answer: \(x(t) = -5\) is unstable. \(x(t) = -3\) is stable. \(x(t) = 4\) is unstable.

4. Consider the differential equation

\[ \frac{\partial}{\partial t} x(t) = \frac{x^3 - 6x^2 - 25x + 150}{x + 3}. \]

(a) Are there any places where Theorem 2.4.2 does not apply? Answer: There is a singularity at \(x = -3\) where Theorem 2.4.2 does not apply.

(b) Find all stationary solutions. Answer: The stationary roots are \([-5, 5, 6]\).
c) For each stationary solution, determine if it is stable, unstable, semistable. Answer:

\( x(t) = -5 \) is stable. \( x(t) = 5 \) is stable. \( x(t) = 6 \) is unstable.

5. Consider the differential equation

\[-144 x(t) - 26 \frac{\partial}{\partial t} x(t) - \frac{\partial^2}{\partial^2 t} x(t) = 0.\]

(a) Find the characteristic equation. Answer:

\[-r^2 - 26r - 144 = 0\]

(b) Find a fundamental solution set. Answer:

\( \{ e^{-8t}, e^{-6t} \} \)

(c) Construct the general solution. Answer:

\[ x(t) = C_1 e^{-8t} + C_2 e^{-6t} \]

6. Describe the behavior of the solutions of the differential equation \( y' = \cos(y) + 1 \). Answer: This is an autonomous 1st order nonlinear ordinary differential equation. The steady-state solutions of this equation are of the form \( y(x) = n\pi \), where \( n \) is an odd integer. Starting at any initial condition, the solution will increase monotonically up to the nearest steady-state solution.

7. Consider the differential equation

\[ 68 x(t) + 13 \frac{\partial}{\partial t} x(t) - \frac{\partial^2}{\partial^2 t} x(t) = 0. \]

(a) Find the characteristic equation. Answer:

\[-r^2 + 13r + 68 = 0\]

(b) Find a fundamental solution set. Answer:

\( \{ e^{-4t}, e^{17t} \} \)

(c) Construct the general solution. Answer:

\[ x(t) = C_1 e^{-4t} + C_2 e^{17t} \]

(d) Find the specific solution satisfying initial conditions \( x(0) = -3 \), \( \dot{x}(0) = 75 \). Answer:

\[ x(t) = -6e^{-4t} + 3e^{17t} \]

8. Consider the differential equation

\[ 40 x(t) + 3 \frac{\partial}{\partial t} x(t) - \frac{\partial^2}{\partial^2 t} x(t) = 0. \]

(a) Find the characteristic equation. Answer:

\[-r^2 + 3r + 40 = 0\]

(b) Find a fundamental solution set. Answer:

\( \{ e^{8t}, e^{-5t} \} \)
(c) Construct the general solution. Answer:
\[ x(t) = C_1e^{8t} + C_2e^{-5t} \]

(d) Find the specific solution satisfying initial conditions \( x(0) = 7, \dot{x}(0) = -9 \). Answer:
\[ x(t) = 2e^{8t} + 5e^{-5t} \]

9. Consider the function
\[ y(x; a, b) = ae^{3x} + be^{-3x}. \]

(a) Find a differential equation for \( y(x) \) that is linear, second order, autonomous, and independent of \( a \) and \( b \). Answer: \( y'' - 9y = 0 \)

(b) If \( y(0) = 0, y'(0) = 12 \), find \( a \) and \( b \). Answer: \( a = 2, b = -2 \).

(c) If \( y(0) = 1, y'(0) = 0 \), find \( a \) and \( b \). Answer: \( a = 1/2, b = 1/2 \).