Homework 3, Math 250-07

due Friday, September 21, 2012

This home covers material from Sections 2.5, and 3.1. Refer there for more practice and help. I urge you to do as many problems in the textbook as possible in addition to this assignment.

1. Consider the differential equation

$$\frac{\partial}{\partial t} \mathrm{x}(t) = -x^2 - x + 6.$$

   (a) Are there any places where Theorem 2.4.2 does not apply? Answer: *This is a nonlinear autonomous equation. The right hand side is continuous everywhere, so Theorem 2.4.2 always applies.*

   (b) Find all stationary solutions. Answer: *The set of stationary roots is* $\{2, -3\}$.

   (c) For each stationary solution, determine if it is stable, unstable, semistable. Answer: $x(t) = -3$ *is unstable.* $x(t) = 2$ *is stable.*

   (d) For each possible initial condition, determine the asymptotic behavior of the corresponding solution. Answer: *If* $x(0) \in (-\infty, -3)$, *then* $x(t)$ *diverges to* $-\infty$. *If* $x(0) \in (-3, \infty)$, *then* $x(t)$ *converges to to* 2. *If* $x(0) = -3$, *then* $x(t) = -3$ *always.*

2. Consider the differential equation

$$\frac{\partial}{\partial t} \mathrm{x}(t) = -x^2 - 16x - 64.$$

   (a) Are there any places where Theorem 2.4.2 does not apply? Answer: *This is a nonlinear autonomous equation. The right hand side is continuous everywhere, so Theorem 2.4.2 always applies.*

   (b) Find all stationary solutions. Answer: *The set of stationary roots is* $\{-8\}$.

   (c) For each stationary solution, determine if it is stable, unstable, semistable. Answer: $x(t) = -8$ *is semistable.*

3. Consider the differential equation

$$\frac{\partial}{\partial t} \mathrm{x}(t) = e^{x^3+4x^2-17x-60} - 1.$$

   (a) Are there any places where Theorem 2.4.2 does not apply? Answer: *This is a nonlinear autonomous equation. The right hand side is continuous everywhere, so Theorem 2.4.2 always applies.*

   (b) Find all stationary solutions. Answer: *The stationary roots are* $[-5, -3, 4]$.

   (c) For each stationary solution, determine if it is stable, unstable, semistable. Answer: $x(t) = -5$ *is unstable.* $x(t) = -3$ *is stable.* $x(t) = 4$ *is unstable.*

4. Consider the differential equation

$$\frac{\partial}{\partial t} \mathrm{x}(t) = \frac{x^3 - 6x^2 - 25x + 150}{x + 3}.$$

   (a) Are there any places where Theorem 2.4.2 does not apply? Answer: *There is a singularity at* $x = -3$ *where Theorem 2.4.2 does not apply.*

   (b) Find all stationary solutions. Answer: *The stationary roots are* $[-5, 5, 6]$.

(c) For each stationary solution, determine if it is stable, unstable, semistable. Answer: *$x(t) = -5$ is stable. $x(t) = 5$ is stable. $x(t) = 6$ is unstable.*

5. Consider the differential equation

$$-144\,\mathrm{x}(t) - 26\frac{\partial}{\partial t}\mathrm{x}(t) - \frac{\partial^2}{\partial^2 t}\mathrm{x}(t) = 0.$$

(a) Find the characteristic equation. Answer:

$$-r^2 - 26r - 144 = 0$$

(b) Find a fundamental solution set. Answer:

$$\{e^{-18t}, e^{-8t}\}$$

(c) Construct the general solution. Answer:

$$x(t) = C_1 e^{-18t} + C_2 e^{-8t}$$

6. Describe the behavior of the solutions of the differential equation $y' = \cos(y) + 1$. Answer: *This is an autonomous 1st order nonlinear ordinary differential equation. The steady-state solutions of this equation are of the form $y(x) = n\pi$, where n is an odd integer. Starting at any initial condition, the solution will increase monotonely up to the nearest steady-state solution.*

7. Consider the differential equation

$$68\,\mathrm{x}(t) + 13\frac{\partial}{\partial t}\mathrm{x}(t) - \frac{\partial^2}{\partial^2 t}\mathrm{x}(t) = 0.$$

(a) Find the characteristic equation. Answer:

$$-r^2 + 13r + 68 = 0$$

(b) Find a fundamental solution set. Answer:

$$\{e^{-4t}, e^{17t}\}$$

(c) Construct the general solution. Answer:

$$x(t) = C_1 e^{-4t} + C_2 e^{17t}$$

(d) Find the specific solution satisfying initial conditions $x(0) = -3$, $\dot{x}(0) = 75$. Answer:

$$x(t) = -6e^{-4t} + 3e^{17t}$$

8. Consider the differential equation

$$40\,\mathrm{x}(t) + 3\frac{\partial}{\partial t}\mathrm{x}(t) - \frac{\partial^2}{\partial^2 t}\mathrm{x}(t) = 0.$$

(a) Find the characteristic equation. Answer:

$$-r^2 + 3r + 40 = 0$$

(b) Find a fundamental solution set. Answer:

$$\{e^{8t}, e^{-5t}\}$$

(c) Construct the general solution. Answer:

$$x(t) = C_1 e^{8t} + C_2 e^{-5t}$$

(d) Find the specific solution satisfying initial conditions $x(0) = 7$, $\dot{x}(0) = -9$. Answer:

$$x(t) = 2e^{8t} + 5e^{-5t}$$

9. Consider the function

$$y(x; a, b) = ae^{3x} + be^{-3x}.$$

(a) Find a differential equation for $y(x)$ that is linear, second order, autonomous, and independent of a and b. Answer: $y'' - 9y = 0$

(b) If $y(0) = 0$, $y'(0) = 12$, find a and b. Answer: $a = 2$, $b = -2$.

(c) If $y(0) = 1$, $y'(0) = 0$, find a and b. Answer: $a = 1/2$, $b = 1/2$.