Homework 2, Math 250-07

due Wednesday, September 12, 2012

This home covers material from Sections 2.3, and 2.4. Refer there for more practice and help. I urge you to do as many problems in the textbook as possible in addition to this assignment.

1. A tank is filled with 200 liters of a solution containing 100 grams of salt. A solution containing a concentration of 2 g/liter salt enters the tank at the rate 4 liters/minute and the well-stirred mixture leaves the tank at the same rate. Set up the initial value problem for the amount of salt in the tank at time t, find the particular solution and find the limiting amount of salt in the tank as $t \to \infty$. Answer: *Let $Q(t)$ be the amount of salt in the tank. Translating into mathematics, $Q(0) = 100$ grams and*

$$\dot{Q} = r\left(c_{in} - \frac{Q}{V}\right).$$

where $c_{in} = 2$ g/L, $V = 200$ L, and $r = 4$ L/min, and time t is measured in minutes. We can use the standard answer from class, or if we forget, we can do the long-form solution by integration.

$$\frac{dQ}{Q - c_{in}V} = -\frac{r}{V}dt$$

$$\ln|Q(t) - c_{in}V| = -\frac{r}{V}t + C$$

$$Q(t) = c_{in}V + e^{-\frac{r}{V}t+C}$$

$$Q(0) = c_{in}V + e^{C}$$

$$Q(t) = c_{in}V + (Q(0) - c_{in}V)e^{-\frac{r}{V}t}$$

Now, we apply our known parameter values.

$$Q(t) = 2 \times 200 + (100 - 2 \times 200)e^{-\frac{4}{200}t}$$

$$Q(t) = 400 - 300e^{-\frac{t}{50}}$$

$$\lim_{t\to\infty} Q(t) = 400.$$

2. A tank has 100 liters of water flow in every hour. Water flows out a rate of $110 + 10\sin(2\pi t)$ liters per hour. If there are 200 liters in the tank initially, how much water is in the tank 3 hours later? Answer: $\dot{V} = 100 - 110 - 10\sin(2\pi t)$, $\dot{V} = -10 - 10\sin(2\pi t)$

$$V(t) = C - 10t + \frac{10}{2\pi}\cos(2\pi t).$$

Applying the initial condition,

$$V(t) = 200 - 10t + \frac{10}{2\pi}\left(\cos(2\pi t) - 1\right).$$

$$V(3) = 200 - 30 + \frac{10}{2\pi}\left(\cos(6\pi) - 1\right) = 170.$$

3. Particulate nitrogen is deposited in a 28,800 gallon pool at a range of $80 + 30\cos(t)$ grams per day. The pool filter pumps 10 gallons per minute, and removes 20 percent of the nitrogen passing through. Right after being filled, the pool contains 400 grams of particulate nitrogen. Determine the amount of nitrogen in the pool over time, starting from the first day the pool is filled. Then determine the amount of particulate nitrogen that accumulates in the filter over time. Answer: *When we set things up, we have to convert everything to common units. In this case, I choose grams.*

$$\dot{Q} = 80 + 30\cos(t) - 60 \times 24 \times 10 \times 0.2 \times \frac{Q}{28,800}$$

$$\dot{Q} = 80 + 30\cos(t) - 0.1Q$$
$$\dot{Q} + 0.1Q = 80 + 30\cos(t)$$
$$Q(t) = e^{-t/10}\left[\int_0^t (80 + 30\cos(u))e^{u/10}du + C\right]$$
$$Q(t) = e^{-t/10}\left[\int_0^t (80 + 30\cos(u))e^{u/10}du + 400\right]$$
$$Q(t) = 800 + \frac{300}{101}\cos(t) + \frac{3000}{101}\sin(t) - \frac{40700}{101}e^{-t/10}$$

The rate at which nitrogen accumulates in the filter is the same as the rate at which it leaves the pool, so if $F(t)$ is the amount in the filter,

$$\dot{F} = .1Q$$
$$F(t) = \frac{1}{10}\int_0^t Q(t)dt = 80t + \frac{30}{101}\sin(t) - \frac{300}{101}\cos(t) - 400 + \frac{40700}{101}e^{-t/10}$$

4. Both $y(x) = 0$ and $y(x) = x^2$ are solutions of the linear first-order equation $xy' - 2y = 0$ passing through the point $y(0) = 0$. But Theorem 2.4.1 says there should exist one and only one solution of a linear first-order ODE. How do we reconcile this apparent contradiction? Answer: *In standard form, this ODE is $y' - 2y/x = 0$. Now, Theorem 2.4.1 says there is a unique solution as long as $y' + p(x)y = g(x)$ where $p(x)$ and $g(x)$ are continuous around the point. However, $2/x$ is not continuous at $x = 0$! So theorem 2.4.1 does not apply at $y(0) = 0$. and there is no disagreement.*

5. Suppose $y(0) = 1$ and $y' - y^2 = 0$. Theorem 2.4.2 says there is some interval around $x = 1$ [Correction: $x = 0$, the $x = 1$ was a typo.] where we can find a solution passing through the initial point. If our solution has to be continuous, what's the largest such interval where our solution will be defined. (Hint: If you get stuck, try graphing your solution.) Answer: *When we calculate the solution, we find $y(x) = 1/(1 - x)$. This is continuous up to $x = 1$, so our interval is $(-\infty, 1)$.*

6. Given
$$\sin(x)\frac{dy}{dx} + \frac{y}{x - 6} = x^3, \quad y(4) = 2,$$
what's the largest interval around our point where we know we can find a solution? Answer: *This is a linear equation. In standard form,*
$$\frac{dy}{dx} + \frac{y}{\sin(x)(x - 6)} = \frac{x^3}{\sin(x)}$$
We see that there are singularities at $x \in \pi\mathbb{Z}$ and $x = 6$. Since we start at $x = 4$, the largest interval is $x \in (\pi, 6)$ (which contains 4).