Homework 1, Math 250-07

Wednesday, September 12, 2012

This home covers material from Sections 1.1, 1.2, 1.3, 2.2, and 2.1. Refer there for more practice and help.

1. Classify the following differential equations as ODE or PDE[1], autonomous or non-autonomous. If the equation is an ODE, determine it's order and if the equation is separable, ratio-homogeneous, linear, or other. If an equation is a 1st-order ordinary differential equation and separable, ratio-homogeneous, or linear[2], find the general solution by using the appropriate integration method. Otherwise, state the function you would try to solve for but DO NOT attempt to solve the other equations.

   (a)
   $$\frac{dy}{dx} = x^2 - 10\sqrt{x}$$
   Answer:*ODE, 1st order, separable, linear, non-autonomous.*
   $$y(x) = x^3/3 - \frac{20}{3}\sqrt{x^3} + C$$

   (b)
   $$x^3 - \frac{da}{dx} = 0$$
   Answer: *ODE, 1st order, separable, linear, non-autonomous.* $a(x) = x^4/4 + C$.

   (c)
   $$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} - 2 = 0$$
   Answer: *PDE, 2nd order, separable, linear, autonomous.* $\Phi(x, t)$.

   (d)
   $$y'' + xy' - y = \sin(x)$$
   Answer: *ODE, 2st order, separable, linear, non-autonomous, variable coefficient.* $y(x) = x^4/4 + C$.

   (e)
   $$\dddot{x} + 4yt\dot{x} = 0$$
   Answer: *ODE, 3rd order, linear, non-autonomous. Find* $x(t)$ *(or* $x(t, y)$ *-* y *is an independent variable).*

   (f)
   $$3\cos(3x) + (10w - 4)w' = 0$$
   Answer: *1st order ODE, separable, non-autonomous.*
   $$\sin(3x) + 5y(x)^2 - 4y(x) = C$$

[1] just a reminder, ODE is an acronym for Ordinary differential equation, and PDE is an acronym for Partial Differential Equation

[2] note that some linear equations are separable, but some are not.

(g)

$$\frac{dy}{dx} - \frac{y}{x} + e^{y/x} = 0$$

Answer: *ODE, 1st order, ratio-homogeneous, let* $ux = y$, $dy = udx + xdu$,

$$u + xdu/dx - u + e^u = 0$$
$$xdu/dx = -e^u$$
$$e^{-u}du = \frac{-dx}{x}$$
$$-e^{-u} = -\ln x + C$$
$$u(x) = -\ln(\ln x - C)$$
$$y(x) = -x\ln(\ln x - C)$$

(h)

$$\frac{dx}{dt} - 5x = te^{-t}$$

Answer: *ODE, 1st order, linear, non-autonomous To solve, look for an integrating factor* $m(t)$ *such that* $-5m = dm/dt$. *So* $m(t) = e^{-5t}$.

$$\dot{x} - 5x = te^{-t}$$
$$m\dot{x} - 5xm = mte^{-t}$$
$$\frac{d}{dt}(e^{-5t}x(t)) = te^{-6t}$$
$$e^{-5t}x(t) = \int te^{-6t}dt + C$$
$$x(t) = (t/6 - 1/36)\,e^{-t} + C$$

(i)

$$\frac{1}{y'} + \frac{1}{x} = 13$$

Answer: *1st order linear separable ODE.*

$$\int \frac{xdx}{13x - 1} = \int dy$$
$$y(x) = \frac{1}{169}(13x - 1 - \ln|13x - 1|) + C$$

(j)

$$i\frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial x^2} + x^2\Psi$$

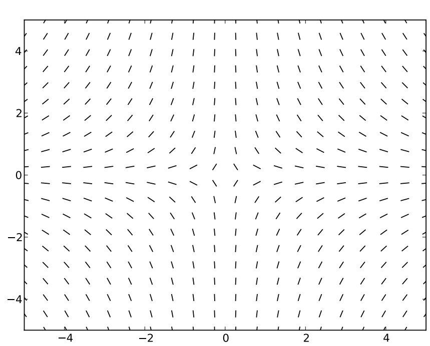
Answer: *This is a PDE, linear, non-autonomous, and we solve for* $\Psi(t, x)$. *This is a Schrdinger equation.*

2. Find a specific solution $v(t)$ to the equation $\dot{v} = 6 - 3v$ that passes through the initial point $v(0) = 3$. Answer: *Solvable by separation or with an integrating factor. The general solution is* $v(t) = 2 + Ce^{-3t}$. *if* $v(0) = 3$, *then* $3 = 2 + C$, *so* $C = 1$, *and our specific solution is* $v(t) = 2 + e^{-3t}$.

3. Draw a slope field for the ordinary differential equation

$$\frac{dy}{dx} = -2\frac{y}{x}$$

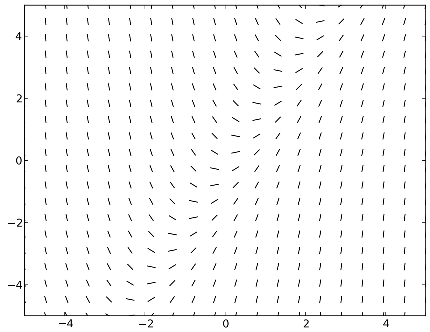
Answer:



4. Draw a slope field for the 1st-order ordinary differential equation

$$\frac{dy}{dx} = 2x - y$$

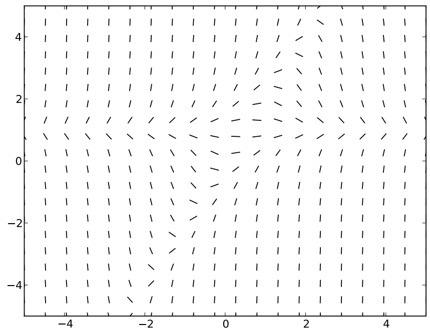
Answer:



5. Draw a slope field for the 1st-order ordinary differential equation

$$\frac{dy}{dx} = (-2x + y)(y - 1)$$

Answer:



6. Check if the curve $f(t) = t^2 + 1$ is a solution of each of the following ODE's.

   (a) $\ddot{f} = 0$ Answer:*no*
   (b) $\ddot{f} = 2$ Answer:*yes*
   (c) $\dot{f} = 1$ Answer:*no*
   (d) $\dot{f} = f$ Answer:*no*
   (e) $\ddot{f} + \dot{f} = 1$ Answer:*no*
   (f) $t\ddot{f} = \dot{f}$ Answer:*yes*
   (g) $t\dot{f} - 2f = 1$ Answer:*no*
   (h) $\ddot{f}^2(t + f) = (\dot{f} - 1)^2 + 3$ Answer:*no*

Challenge: What does it really mean for something to be a solution of an ordinary differential equation? Here's a problem that will help you develop some intuition. Graphing the problem as you go may help.

Suppose we have equation $(y')^2 + 4y - 4 = 0$, where $y(x)$.

- Show that $y(x) = 1$ is a solution.
- Show that $y(x) = 1 + x^2$ is not a solution.
- Show that $y(x) = 1 - x^2$ is a solution.

- Show that $y(x) = \max\{1 - (x-1)^2, 1 - (x+1)^2\}$ is not a global solution by finding a coordinate x where the solution fails. Answer: *This function has a corner, aka a ''cusp'', at $x = 0$. You can see this if you plot it out. The derivative is not defined at this corner because different limiting processes give different results.*

$$\lim_{h\to 0} \frac{y(0+h) - y(0-h)}{2h} = \frac{[1-(h-1)^2] - [1-(-h+1)^2]}{2h} = \frac{(h-1)^2 - (-h+1)^2}{2h} = 0$$

$$\lim_{h\to 0} \frac{y(0+h) - y(0)}{h} = \frac{[1-(h-1)^2] - [0]}{2h} = \frac{1-(h^2-2h+1)}{2h} = \frac{2h-h^2}{2h} = 1.$$

  These are two different numbers, implying the derivative is undefined. Without out a derivative, we can not satisfy our ODE at $x = 0$.

- Explain why

$$y(x) = \begin{cases} 1-(x-1)^2 & \text{if } x \geq 1, \\ 1-(x+1)^2 & \text{if } x \leq -1, \\ 1 & \text{otherwise} \end{cases}$$

  does not suffer from the problem of the previous solution. Answer: *On the other hand, while there are transitions in this function at $x = 1$ and $x = -1$, these transitions are smooth -- there are no corners, and the solution is well-defined.*