Projective connections, group Vey cocycle and deformation quantization

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0. Introduction

0.1. In this paper we construct a number of geometrically meaningful 1– and 2–cocycles of infinite dimensional Lie groups and Lie algebras. The groups are groups of diffeomorphisms of smooth manifolds and their subgroups, such as the group of symplectomorphisms of a symplectic manifold; the Lie algebras are algebras of vector fields. Coefficient spaces of the cocycles are spaces of various tensor fields, naturally acted upon by diffeomorphisms and vector fields. Our constructions are closely related to deformation quantization.

0.2. Definition of cohomology. Recall the definition of Lie group and Lie algebra cohomology (see, e.g., [Fu]). For a group G and its left module A the space of cochains $C^q(G,A)$ consists of continuous maps $\underbrace{G \times ... \times G}_{q} \to A$, and the differential $d : C^q(G,A) \to C^{q+1}(G,A)$ is given by the formula:

$$dC(g_1,...,g_{q+1}) = g_1 C(g_2,...,g_{q+1}) \ + \sum_{1\le i\le q} (-1)^i C(g_1,...,g_i g_{i+1},...,g_{q+1}) +$$

$$(-1)^{q+1} C(g_1,...,g_q).$$

For a Lie algebra g and its module A the space of cochains $C^q(\mathrm{g},A)$ is $Hom(\wedge^q \mathrm{g}, A)$, and the differential is given by the formula:

$$dc(g_1,...,g_{q+1}) = \sum_{1\le i<j\le q+1} (-1)^{i+j-1} c([g_i,g_j],g_1,...,\hat{g}_i,...,\hat{g}_j,...,g_{q+1}) +$$

$$\sum_{1\le i\le q+1} (-1)^i g_i c(g_1,...,\hat{g}_i,...,g_{q+1}).$$

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There is a natural homomorphism $H^q(G, A) \to H^q(\mathrm{g}, A)$, and throughout the paper we use upper case characters for Lie group cocycles and the same lower case ones for the corresponding cocycles of Lie algebras.

0.3. Simple example. The following instructive construction serves a model for more involved constructions of the paper.

Let (M, μ) be a manifold with a volume form. Assign to an orientation preserving diffeomorphism f a smooth function $\bar{C}(f)$ by

$$f^*\mu = e^{\bar{C}(f)}\mu.$$

Set: $C(f) = \bar{C}(f^{-1})$. This map $C : Diff_+M \to C^\infty(M)$ is a 1-cocycle :

$$C(fg) = C(f) + fC(g);$$

here $f\phi$ is the action of the diffeomorphism f on a function ϕ:

$$(f\phi)(x) = \phi(f^{-1}(x)).$$

The corresponding Lie algebra cocycle $c : VectM \to C^\infty(M)$ is the divergence operator.

0.4. Organization of the paper and acknowledgements. Section 1 contains main constructions and results. Further details and proofs are given in Section 2.

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1. Main constructions and results

1.1. Triangle area construction. Given a (germ of a) diffeomorphism f of the affine plane V, how can one measure its failure to be projective, that is, to send straight lines to straight lines? The following construction, which I learned from E. Ghys and which was inspired by the book [C-F], provides a natural answer.

Use affine structure to identify V with its tangent plane at every point. Let $x \in V$ be a point and u a tangent vector at x. Consider the three collinear points $(x - \epsilon u, x, x + \epsilon u)$, where ϵ is a small parameter, and apply f to this triple. One obtains a triangle whose side lengths are of order ϵ and whose vertex angle is ϵ-close to π. It follows that the oriented area of the triangle is of order ϵ^3. Divide by ϵ^3, take limit $\epsilon \to 0$, and denote the resulting number by $\bar{A}_{(x,u)}(f)$.

For a fixed f and x, by construction, $\bar{A}_{(x,u)}(f)$ is a cubic form on the tangent space T_xV. Hence $\bar{A}$ may be thought of as a map

$$\bar{A} : Diff\ V \to Sect\ S^3(T^*V).$$

This map depends on the 2-jet of a diffeomorphism and vanishes if the diffeomorphism is projective. Here is an explicit formula:

$$\bar{A}(f) = [f_x, f_{xx}]dx^3 + (2[f_x, f_{xy}] + [f_y, f_{xx}])dx^2dy + (2[f_y, f_{xy}] + [f_x, f_{yy}])dxdy^2 + [f_y, f_{yy}]dy^3,$$

where $f(x, y)$ is a vector function and $[\ ,\ ]$ is the determinant.

The behavior of $\bar{A}$ under composition of diffeomorphisms is as follows:

$$\bar{A}(fg) = J(f)\bar{A}(g) + g^*\bar{A}(f),$$

where $g^*\bar{A}(f)$ is the induced differential form, and $J(f)$ is the Jacobian. In the case of most interest to us, when f is area preserving, one has:

$$\bar{A}(f) = [f_x, f_{xx}]\ dx^3 + 3[f_x, f_{xy}]\ dx^2dy + 3[f_y, f_{xy}]\ dxdy^2 + [f_y, f_{yy}]\ dy^3,$$

and

$$\bar{A}(fg) = \bar{A}(g) + g^*\bar{A}(f).$$

This implies that the map

$$A : Diff\ V \to Sect\ S^3(T^*V);\quad A(f) = \bar{A}(f^{-1})$$

is a 1-cocycle of the group of symplectomorphisms with coefficients in cubic differentials.

The corresponding 1-cocycle of the Lie algebra of Hamiltonian vector fields is:

$$a(sgrad\ f) = f_{xxx}\ dx^3 + 3f_{xxy}\ dx^2dy + 3f_{xyy}\ dxdy^2 + f_{yyy}\ dy^3,$$

where $sgrad\ f$ is the symplectic gradient of a function $f(x, y)$.

The triangle area construction extends verbatum to the case when f is a symplectomorphism of linear symplectic space V^{2n}: one uses the symplectic form to measure areas of triangles. In this more general setting the next result holds.

Theorem. *$A \in C^1(SDiff\ V, Sect\ S^3(T^*V))$ is a 1-cocycle, not homologous to zero.*

1.2. Schwarzian derivative. The above construction is analogous to that of the Schwarzian derivative. Given a diffeomorphism f of the projective line, a point $x \in \mathbf{RP}^1$ and a tangent vector $u \in T_x\mathbf{RP}^1 = \mathbf{R}^1$, consider the cross-ratio

$$[f(x),\ f(x+\epsilon u),\ f(x+2\epsilon u),\ f(x+3\epsilon u)].$$

This cross-ratio does not contain linear terms in ϵ. The coefficient at ϵ^2 is a quadratic differential on $\mathbf{RP}^1$ which, up to a constant factor, equals the Schwarzian derivative

$$S(f) = \Big[\frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2\Big]\ dx^2.$$

The Schwarzian derivative is an obstruction to f being projective, and it enjoys the property:

$$S(fg) = S(g) + g^*S(f).$$

Note that, unlike the cocycle $A(f)$, the Schwarzian derivative depends on the 3-jet of a diffeomorphism.

1.3. Two Lie algebra homomorphisms $P(V) \to Sect\ S^*(TV)$. In this section we identify the Lie algebra 1-cocycle a, corresponding to the group cocycle A from 1.1, as an infinitesimal deformation of an imbedding of Lie algebras.

Denote by $P(M)$ the Poisson Lie algebra of smooth functions on a symplectic manifold M. The algebra $P(M)/Const$ is the Lie algebra of the Lie group $SDiff\ M$.

The Lie algebra 1-cocycle

$$a \in C^1(P(V), Sect\ S^3(T^*V))$$

vanishes on constants. Using the isomorphism $V^* = V$, provided by the symplectic structure, one imbeds $Sect\ S^3(T^*V)$ to the space of symmetric tensor fields $Sect\ S^*(TV)$. Thus a may be considered as a 1-cocycle with values in $Sect\ S^*(TV)$.

For an arbitrary smooth manifold M the space $Sect\ S^*(TM)$ is a Lie algebra with respect to the, so called, symmetric Schouten concomitant. Under the identification of symmetric tensor fields with fiberwise polynomial functions on T^*M, this operation is the Poisson bracket. In particular, vector fields are fiberwise linear functions on T^*M.

If M is a symplectic manifold then one has an imbedding of Lie algebras:

$$sgrad : P(M)/Const \to Sect\ S^*(TM). \qquad (*)$$

Given a pair of Lie algebras h $\subset$ g, the space $H^1(\mathrm{h},\mathrm{g})$ is the space of equivalence classes of infinitesimal deformations of the imbedding h $\subset$ g – see [O-T]. If a formal imbedding $i_t : \mathrm{h} \to \mathrm{g}$ is given by

$$i_t(g) = g + t\alpha_1(g) + t^2\alpha_2(g) + ...,$$

where $\alpha_i : \mathrm{h} \to \mathrm{g}$ are linear maps, then α_1 is a 1-cocycle in $C^1(\mathrm{h},\mathrm{g})$. We apply this consideration to the imbedding (*).

Let (x, y) be Darboux coordinates in V^{2n} and (u, v) the dual basis of V. Define the differential operators D_k: for $f(x, y) \in C^\infty(V)$ let

$$D_k(f) = (u\partial_y - v\partial_x)^k(f) = \sum_{|I|+|J|=k} (-1)^{|I|} \frac{k!}{I!J!}\, \frac{\partial^k f}{\partial_{x_I}\partial_{y_J}} u^J v^I \in Sect\ (S^kTV);$$

here I and J are multiindices, and $I! = (i_1, ..., i_n)! = i_1!...i_n!$. In particular, $D_0(f) = f$ and $D_1(f) = sgrad\ f$.

Consider the formal linear maps $P(V) \to Sect\ S^*(TV)$:

$$I_t(f) = \frac{1}{2}\sum_{k\geq 0} \frac{D_k(f)}{k!} t^{k-1} = \frac{1}{2t}\ \exp\ (tD_1)(f),$$

$$J_t(f) = \sum_{k\geq 0} \frac{D_{2k+1}(f)}{(2k+1)!} t^k = \frac{1}{\sqrt{t}}\ \sinh\ (\sqrt{t}D_1)(f).$$

Note that $J_{t^2} = I_t + I_{-t}$.

Theorem. *1). The cocycle a, considered as a map $P(V) \to S^3(TV)$, is, up to a constant multiplier, the operator D_3. 2). The maps I_t and J_t are Lie algebra homomorphisms.*

In particular, J_t is a formal deformation of the imbedding (*), and the corresponding infinitesimal deformation is the cocycle a.

Similar homomorphisms

$$I_t \ \text{ and } \ J_t : P(M) \to Sect\ S^*(TM) \subset P(T^*M)$$

will be constructed for every symplectic manifold M in Section 2.5.

1.4. 2-jet construction and the cocycle B. Another, a more systematic, way to construct obstructions to projectivity is to consider the left action of 2-jets of projective diffeomorphisms on 2-jets of diffeomorphisms.

Given a germ of a diffeomorphism f of linear space V^n, $n \geq 2$, with the source x and the target y, compose it with germs of projective maps h with the source y and the target

x. One can find h such that $j^1(hf) = j^1(Id_x)$. The space of 2-jets of diffeomorphisms ϕ with $j^1(\phi) = j^1(Id_x)$ identifies with the space of componentwise quadratic maps $S^2V^* \otimes V$. Consider the natural pairing:

$$div : S^2V^* \otimes V \to V^*; \text{ set : } T = Ker\ div$$

(if one interprets $S^2V^* \otimes V$ as componentwise quadratic vector fields then div is the divergence). Then there exists a unique projective h such that $j^1(hf) = j^1(Id_x)$ and the quadratic part of hf, which we denote by $\bar{B}(f)$, belongs to T. Let $B(f) = \bar{B}(f^{-1})$. We have assigned a tensor field $B(f)$ of type T to a diffeomorphism f.

Theorem. *B is a nontrivial 1-cocycle of the group $Diff\ V$ with coefficients in the space of tensor fields $Sect\ T$.*

The construction of the cocycle B extends to an arbitrary manifold M. This involves a choice of a projective connection on M, but the cohomology class of the resulting cocycle $B \in C^1(Diff\ M, Sect\ T)$ does not depend on this choice. The details of this construction are given in Section 2.3.

1.5. Relation between the cocycles A and B. In the case of linear symplectic space the relation between the cocycle $A \in C^1(SDiff\ V, Sect\ S^3(T^*V))$ from 1.1 and the restriction of the cocycle $B \in C^1(Diff\ M, Sect\ T)$ from 1.4 to the group $SDiff\ V$ is as follows.

The isomorphism $V^* = V$, along with the natural projection $S^2V^* \otimes V^* \to S^3V^*$, provides the map

$$\pi : T \subset S^2V^* \otimes V = S^2V^* \otimes V^* \to S^3V^*.$$

Denote by the same symbol the map of the spaces of sections.

Theorem. *One has: $A = \pi B$.*

1.6. 2-cocycles from 1-cocycles. There is a general procedure to manufacture 2-cocycles from 1-cocycles. Let G be a group, U and W its modules, $\nu : \wedge^2 U \to W$ a G-homomorphism and $C \in C^1(G, U)$ a 1-cocycle. Let

$$D(g_1, g_2) = \nu(C(g_1), C(g_1 g_2)).$$

Similarly, for a Lie algebra g, a g-homomorphism $\nu : \wedge^2 U \to W$ and a 1-cocycle $c \in C^1(\mathrm{g}, U)$, let

$$d(g_1, g_2) = \nu(c(g_1), c(g_2)).$$

Lemma. *The cochains $D \in C^2(G, W)$ and $d \in C^2(\mathrm{g}, W)$ are 2-cocycles, and their cohomology classes depend only on those of C and c. If c is a Lie algebra cocycle corresponding to a Lie group cocycle C then the same holds true for the 2-cocycles d and D.*

Being applied to the 1-cocycle C from 0.3 with $M = S^1$ and

$$\nu : \wedge^2 C^\infty(S^1) \to \mathbf{R} \ \text{ given by } \ \nu(f, g) = \int_{S^1} f \; dg,$$

this yields the famous Bott-Thurston cocycle $D \in C^2(Diff_+ \; S^1)$ – see [B]. The corresponding 2-cocycle of the Lie algebra $Vect \; S^1$ is given by the formula:

$$d(f_1(x) \; \partial_x, f_2(x) \; \partial_x) = \int_{S^1} \begin{vmatrix} f_1'(x) & f_2'(x) \\ f_1''(x) & f_2''(x) \end{vmatrix} \; dx;$$

the class $[d]$ is a generator of $H^2(Vect \; S^1)$.

1.7. Group Vey cocycle. Apply the construction 1.6 to the cocycle $B \in C^1(Diff \; M, Sect \; T)$ from 1.4. Let

$$\nu : \wedge^2 T \subset (S^2 V^* \otimes V)^{\wedge 2} \to \wedge^2 V^*$$

be given by the formula:

$$\nu : (\xi_1 \otimes v_1) \wedge (\xi_2 \otimes v_2) \to div(\xi_1 \otimes v_2) \wedge div(\xi_2 \otimes v_1);$$

here $\xi_{1,2} \in S^2 V^*, \; v_{1,2} \in V$ and div is the same as in 1.4. The map ν induces a $Diff \; M$-invariant map of the spaces of sections, and by 1.6 one obtains from B a group 2-cocycle $\bar{C} \in C^2(Diff \; M, \Omega^2(M))$.

If M is a symplectic manifold, one has a $SDiff \; M$-invariant map $\Omega^2(M) \to C^\infty(M)$, the convolution with the bivector field, dual to the symplectic structure. One obtains from $\bar{C}$ a 2-cocycle

$$C \in C^2(SDiff \; M, C^\infty(M))$$

which we call the group Vey cocycle. The reason for this name is as follows.

Consider the Lie algebra cocycle $c \in C^2(P(M), P(M))$, corresponding to C. It is known that

$$H^2(P(M), P(M)) = H^2_{DR}(M) \oplus \mathbf{R}$$

for every symplectic manifold – see, e.g., [O-R] and the references therein. The one dimensional subspace is generated by a remarkable cohomology class, discovered by J. Vey – see [V]. The Vey class is of great importance in deformation quantization – see next subsection.

The significance of the 2-cocycle c is as follows.

Theorem. *The cohomology class* $[c] \in H^2(P(M), P(M))$ *generates the Vey direct summand* $\mathbf{R} \subset H^2(P(M), P(M))$.

Thus C integrates the Vey cocycle of the Lie algebra $P(M)$ to a 2-cocycle of the group $SDiff\ M$.

1.8. Square roots of the Moyal bracket and of the $\star$-product. Recall that $H^2(\mathrm{g}, \mathrm{g})$ is the space of equivalence classes of infinitesimal deformations of a Lie algebra g (see, e.g., [Fu]). The infinitesimal deformation of the Poisson algebra of an arbitrary symplectic manifold, corresponding to the Vey class, extends to a nontrivial formal deformation of this algebra.

More specifically, there exists a deformation of the associative algebra structure in $C^\infty(M)$ given by a formal series in variable t. It is called $\star$-product and written as $f \star_t g$. The deformation of the Poisson bracket is related to the $\star$-product as follows:

$$\{f, g\}_t = \frac{1}{2\sqrt{t}}\,(f \star_{\sqrt{t}} g - g \star_{\sqrt{t}} f).$$

The deformation quantization theory attracted much attention in recent years – see, e.g., [D], [F-S], [Fe], [O-R], [W].

In the case of linear symplectic space these deformations are given by explicit formulas. We write them in Darboux coordinates (x, y). Define bidifferential operators on functions in (x, y):

$$P_k(f, g) = \sum_{|I|+|J|=k} (-1)^{|J|} \frac{k!}{I!J!}\, \frac{\partial^k f}{\partial_{x_I} \partial_{y_J}}\, \frac{\partial^k g}{\partial_{x_J} \partial_{y_I}}.$$

In particular, $P_0(f, g) = fg$ and $P_1(f, g) = \{f, g\}$, the usual Poisson bracket. One has:

$$f \star_t g = \sum_{k \geq 0} \frac{P_k(f, g)}{k!} t^k; \quad \{f, g\}_t = \sum_{k \geq 0} \frac{P_{2k+1}(f, g)}{(2k+1)!} t^k.$$

The latter operation is called the Moyal bracket.

Notice the striking similarity between these formulas and the ones in 1.3. As we explain below, the homomorphisms I_t and J_t are, in a sense, square roots of the $\star$-product and of the Moyal bracket, respectively.

Extend the symplectic structure to the pairing

$$<\ ,\ >: S^k V \otimes S^k V \to \mathbf{R},$$

which in the basis (u, v), dual to (x, y), writes:

$$< u^I v^J, u^K v^L >= (-1)^{|J|} I!J! \ \text{ if } \ K = J,\ L = I$$

and zero otherwise; here I, J, K, L are multiindices. Denote by the same symbol the pairing on sections:

$$< \, , \, >: (Sect\ S^k(TV))^{\otimes 2} \to C^\infty(V).$$

Extend the pairing to all symmetric tensor fields assuming tensors of different degrees to be orthogonal.

Theorem. *The following identities hold:*

$$< I_s(f), I_t(g) >= \frac{1}{4st}\ f \star_{st} g; \quad < J_s(f), J_t(g) >= \{f, g\}_{st}.$$

The associativity and Jacobi identities for the $\star$-product and the Moyal bracket imply therefore similar identities for the homomorphisms I_t and J_t.

Corollary. *One has:*

$$< I_t(< I_t(f), I_t(g) >), I_t(h) >=< I_t(f), I_t(< I_t(g), I_t(h) >) >,$$

$$< J_t(< J_t(f), J_t(g) >), J_t(h) > + < J_t(< J_t(g), J_t(h) >), J_t(f) > +$$

$$< J_t(< J_t(h), J_t(f) >), J_t(g) >= 0.$$

The role of the homomorphisms I_t and J_t in deformation quantization deserves further study. We finish Section 1 with the following

Conjecture. *For every symplectic manifold M there exist Lie algebra homomorphisms*

$$I_t \ \text{ and } \ J_t : P(M) \to Sect\ (S^*TM) \subset P(T^*M),$$

given by formal power series in t and satisfying the identities from the above Corollary.

2. Proofs and further constructions

2.1. Cocycle B: the case of linear space. We use the notation from 1.4. Let $G(n)$ be the group of 2-jets of origin preserving diffeomorphisms of V^n, and $H(n)$ its subgroup consisting of 2-jets of projective maps. In coordinates, a typical element of $H(n)$ is given by:

$$X_i = \sum_j a_{ij} x_j + (\sum_k b_k x_k)(\sum_j a_{ij} x_j), \quad \det a_{ij} \neq 0, \quad 1 \leq i, j, k \leq n.$$

Denote by H_{xy} the set of 2-jets of projective maps with the source x and target y.

Recall that we have the pairing $div : S^2V^* \otimes V \to V^*$ and $T = Ker\ div$. We interpret $S^2V^* \otimes V$ as componentwise quadratic maps. Let $F(n) \subset G(n)$ consist of 2-jets with the identical linear part and the quadratic part in T. Clearly, $F(n)$ is an affine space with the underlying linear space T.

Lemma 2.1.1. *$F(n)$ is a normal Abelian subgroup of $G(n)$.*

Proof. If $f_i = Id + t_i,\ t_i \in T,\ i = 1,2$ are two elements of $F(n)$ then $f_1 f_2 = Id + t_1 + t_2$. If $f = Id + t \in F(n)$ and $g \in G(n)$ then $gfg^{-1} = Id + ata^{-1}$ where a is the linear part of g. Since $div\ (ata^{-1}) = div\ t = 0$ one has $gfg^{-1} \in F(n)$, and the lemma is proved.

Consider the left action of $H(n)$ on $G(n)$.

Lemma 2.1.2. *$F(n)$ is a transversal to the orbits of this action: for every $f \in G(n)$ there exists a unique $h \in H(n)$ such that $hf \in F(n)$.*

Proof. The linear part of the desired h must be that of f^{-1}; therefore, without loss of generality, assume that $j^1(f) = j^1(Id)$. Let $q \in S^2V^* \otimes V$ be the quadratic part of f. For $h \in H(n)$ with $j^1(h) = j^1(Id)$ one has: $h = Id + b\ Id$ where $b \in V^*$. Then $hf = Id + b\ Id + q$. This lies in $F(n)$ if and only if $div\ (b\ Id + q) = 0$. By the Euler formula $div\ (b\ Id) = (n+1)b$. Therefore $b = -div\ (q)/(n+1)$. This uniquely determines h.

The above lemma justifies the definition of the map $B : Diff\ V \to Sect\ T$ from 1.4.

Lemma 2.1.3. *B is a 1-cocycle.*

Proof. Let $f, g \in Diff\ V,\ f^{-1}(x) = y$ and $g^{-1}(y) = z$. Let $h_1 \in H_{yx}, h_2 \in H_{zy}$ satisfy:

$$h_1 f^{-1} = Id + t_1 \in F(n),\ h_2 g^{-1} = Id + t_2 \in F(n);\quad t_{1,2} \in T.$$

Let a be the linear part of f^{-1} at x. One has:

$$h_1 h_2 g^{-1} f^{-1} = h_1 (Id + t_2) f^{-1} = Id + t_1 + a^{-1} t_2 a.$$

The last term is the action of f on t_2. Therefore $B(fg) = B(f) + fB(g)$.

To prove Theorem 1.4 it remains to show that B is a nontrivial 1-cocycle. Note that B is a differential operator of second order. This would not be the case if B were exact, that is, if $B(f) = fs - s$ for some $s \in Sect\ T$.

2.2 Comparing the cocycles A and B. Recall that the triangle symplectic area construction 1.1 yields a map

$$\bar{A} : SDiff\ V \to Sect\ S^3(T^*V)$$

for linear symplectic space V.

To prove Theorem 1.5 we need to show that $\bar{A}(f) = \pi \bar{B}(f)$ where π and $\bar{B}$ are the same as in 1.4 and 1.5. By construction,

$$\bar{A}(hf) = \bar{A}(f) \text{ and } \bar{B}(hf) = \bar{B}(f)$$

for every affine symplectic map h. Composing f with an appropriate h one trivializes $j^1(f)$. Therefore, without loss of generality, we assume that f preserves the origin O and $j^1(f) = j^1(Id)$.

Thus $j^2(f) = Id + q$ where $q \in S^2V^* \otimes V$ is the quadratic part of f. Fix Darboux coordinates (x, y). Since f is a symplectomorphism there exists a generating cubic form $H(x, y)$ such that $j^2(f)$ is:

$$X_i = x_i - \frac{\partial H}{\partial y_i},\ Y_i = y_i + \frac{\partial H}{\partial x_i},\ \text{ and } \ q = (-\frac{\partial H}{\partial y_i}, \frac{\partial H}{\partial x_i}).$$

Clearly $div\ q = 0$, so the projection of f to $T = Ker\ div \subset S^2V^* \otimes V$ is q itself. The map $\pi : S^2V^* \otimes V \to S^3V^*$ sends q to

$$\sum x_i \frac{\partial H}{\partial x_i} + y_i \frac{\partial H}{\partial y_i} = 3H,$$

the equality due to the Euler formula. Thus $\pi \bar{B}(f) = 3H$.

On the other hand, a straightforward computation shows that the cubic part of the symplectic area of the triangle $(f(-x, -y), O, f(x, y))$ is equal to

$$\omega\ (Id(x, y),\ q(x, y)) = \sum x_i \frac{\partial H}{\partial x_i} + y_i \frac{\partial H}{\partial y_i} = 3H.$$

Therefore $\bar{A}(f) = 3H$. This proves Theorem 1.5, which, in turn, implies Theorem 1.1

2.3. Projective connections and the cocycle B on an arbitrary manifold. To carry out the construction 1.4 and 2.1 of the cocycle B on an arbitrary manifold M one needs an analog of the group of projective 2-jets $H(n)$. This is provided by a projective connection on M.

Recall the notion of projective connection – see [K]. We use the notation from 1.4 and 2.1. A 2-frame on M^n at point x is the 2-jet of a diffeomorphism $(\mathbf{R}^n, O) \to (M, x)$. The set of 2-frames is a principal bundle over M with the structure group $G(n)$ acting on the right. A projective connection is a reduction of this bundle to the subgroup $H(n)$. Note that projective connections exist on every manifold M: they are in one to one correspondence with sections of a bundle over M whose fiber is the contractible space $G(n)/H(n)$.

Said differently, for every $x \in M$ one has a set P_x of 2-jets of diffeomorphisms $(\mathbf{R}^n, O) \to (M, x)$ on which $H(n)$ acts transitively and freely on the right. Call a 2-jet $h : M \to M$ with the source x and target y projective if $hP_x = P_y$. Denote the set of such projective 2-jets by H_{xy}, and abreviate H_{xx} to H_x. For every $\phi \in P_x$ one has: $H_x = \phi H(n) \phi^{-1}$, and for every $f \in H_{xy}$ one has : $H_{xy} = f H_x = H_y f$.

Let G_x be the set of 2-jets of diffeomorphisms $(M,x) \to (M,x)$, and let $F_x = \phi F(n)\phi^{-1}$ for some $\phi \in P_x$; by Lemma 2.1.1 this does not depend on the choice of ϕ. Then F_x is a transversal to the orbits of the left action of H_x on G_x. As in 1.4 we identify F_x with $T_x = Ker\ div$, where $div : S^2(T^*_xM) \to T^*_xM$ is the natural pairing, and denote by q the projection $G_x \to F_x$.

After these preparations construction of the cocycle B repeats the one in the linear case. Given $f \in Diff\ M$, consider $j^2(f^{-1})$ at $x \in M$. Let $y = f^{-1}(x)$; pick $h \in H_{yx}$ and set: $B(f)(x) = q(hf^{-1})$. This gives the 1-cocycle

$$B : Diff\ M \to Sect\ T$$

promised in 1.4 (the proof of the cocycle property is the same as in 2.1).

To complete this construction we prove the following, not very surprising, result.

Lemma. *The cohomology class $[B] \in H^1(Diff\ M, Sect\ T)$ does not depend on the choice of the projective connection.*

Note that, a priori, the cohomology class in question may vary continuously; therefore the fact that two connections are homotopic does not siffice for a proof.

Proof. Given two projective connections, use bar to indicate that the corresponding object is related to the second one. We will show that

$$\bar{B}(f) - B(f) = fs - s \qquad (**)$$

for some $s \in Sect\ T$.

The two connections are related by a field of 2-jets of diffeomorphisms $g_x : (M,x) \to (M,x)$ such that $\bar{P}_x = g_xP_x$. Clearly, $\bar{H}_{xy} = g_yH_{xy}g_x^{-1}$, and, in particular, $\bar{H}_x = g_xH_xg_x^{-1}$. Notice that $\bar{F}_x = F_x$.

Let us compare the two projections $\bar{q}$ and q from G_x to F_x. Given $\phi \in G_x$ there exists $\bar{h} \in \bar{H}_x$ such that $\phi = \bar{h}\bar{q}(\phi)$. Therefore $\phi = g_xhg_x^{-1}\bar{q}(\phi)$ for some $h \in H_x$. Rewrite this as $g_x^{-1}\phi = hg_x^{-1}\bar{q}(\phi)$. There exist $h', h'' \in H_x$ such that $g_x^{-1}\phi = h'q(g_x^{-1}\phi)$ and $g_x^{-1} = h''q(g_x^{-1})$. Therefore $h'q(g_x^{-1}\phi) = hh''q(g_x^{-1})\bar{q}(\phi)$. The formula for multiplication in $F(n)$ from Lemma 2.1.1 implies that

$$\bar{q}(\phi) = q(g_x^{-1}\phi) - q(g_x^{-1}). \qquad (***)$$

Now we can compare the cocycles $\bar{B}$ and B. Given $j^2(f^{-1})$ at $x \in M$ with $y = f^{-1}(x)$, by definition, $\bar{B}(f) = \bar{q}(\bar{h}f^{-1})$ where $\bar{h} \in \bar{H}_{yx}$. One has: $\bar{h} = g_xhg_y^{-1}$ with $h \in H_{yx}$. Apply (***) to $\phi = g_xhg_y^{-1}f^{-1}$:

$$\bar{B}(f) = q(hg_y^{-1}f^{-1}) - q(g_x^{-1}) = B(fg_y) - B(g_x).$$

By cocycle property, $B(fg_y) = B(f) + fB(g_y)$, and, finally,

$$\bar{B}(f) - B(f) = fB(g_y) - B(g_x).$$

This means that $\bar{B}$ and B are cohomologous, the section s in (**) being equal to $B(g)$.

2.4. Identifying the cocycle $c \in C^2(P(M), P(M))$ as the Vey cocycle. First we deduce an explicit formula for the triangle symplectic area cocycle

$$a \in C^1(P(V), Sect\ S^3(T^*V))$$

where (V, ω) is linear symplectic space. Fix Darboux coordinates (x, y).

Lemma 2.4.1. *Up to a constant multiplier,*

$$a(h) = (\partial_x dx + \partial_y dy)^3(h) = \sum_{|I|+|J|=3} \frac{6}{I!J!} \frac{\partial^3 h}{\partial_{x_I} \partial_{y_J}} dx^I dy^J.$$

Proof. The Lie algebra cocycle is reconstructed from the Lie group one as follows:

$$a(h) = \frac{d}{dt}\Big|_{t=0} A(\exp(t\ sgrad\ h)).$$

Thus one needs to compute the cubic part of the symplectic area of the triangle

$$(f(-u,-v),\ f(O),\ f(u,v)),\ \text{where } f(x,y) = (x + th_y,\ y - th_x),$$

and then take its coefficient at t as the value of the cubic differential on the tangent vector $\xi = (u, v)$. Up to a constant, this equals the cubic part in ξ of $\omega(\xi, \eta)$ where η is the vector

$$(-h_y(\xi) - h_y(-\xi) + 2h_y(O),\ h_x(\xi) + h_x(-\xi) - 2h_x(O)).$$

This yields the desired formula.

Composing the cocycle a with the isomorphism $S^3(T^*V) \to S^3(TV)$, provided by the symplectic structure, one obtains the first statement of Theorem 1.3.

Next, consider the cocycle $b \in C^1(P(V), Sect\ T)$. Let $h(x, y)$ be a cubic form. Then

$$b(h) = \frac{d}{dt}\Big|_{t=0} B(\exp(t\ sgrad\ h)),$$

and this is equal to the quadratic part of the map, i.e., to $sgrad\, h = h_y\partial_x - h_x\partial_y$, considered as a section of $T \subset S^2(T^*V) \otimes V$.

Let M be a symplectic manifold. Recall, 1.7, that the cocycle $c \in C^2(P(M), P(M))$ is constructed from b as follows:

$$c(f, g) =< \nu(b(f), b(g)),\ \tilde{\omega} >,$$

where $\nu : Sect\ \wedge^2 T \to \Omega^2(M)$ was defined in 1.7 and $\tilde{\omega}$ is the bivector field, dual to the symplectic structure. A straightforward computation of the right hand side of this formula in the linear case $M = V$ gives the next result.

Lemma 2.4.2. *If f and g are cubic forms then $c(f,g)$ equals, up to a constant multiplier, the differential operator $P_3(f,g)$ from 1.8.*

Since c is a third order homogeneous bidifferential operator, one concludes that, up to a multiplier, $c = P_3$.

Now we can prove Theorem 1.7. The Vey class in $H^2(P(M), P(M))$ is uniquely determined by the following two properties: it is given by a local cocycle, that is, by a bidifferential operator, and in Darboux coordinates the cocycle is P_3 – see [O-R]. A multiple of the class $[c]$ has both properties, therefore $[c]$ is proportional to the Vey class.

2.5. Properties of the homomorphisms I_t and J_t. In this last section we prove the second statement of Theorem 1.3, construct homomorphisms I_t and J_t on an arbitrary symplectic manifold and prove Theorem 1.8.

Start with Theorem 1.3. Let (x,y) be Darboux coordinates in symplectic space (V,ω) and (u,v) the dual basis of V. Then (x,y,u,v) are Darboux coordinates in T^*V with its canonical symplectic structure $\Omega = dx \wedge du + dy \wedge dv$. Consider the new coordinates in T^*V:

$$p = x - tv,\ q = y + tu,\ \bar{p} = x + tv,\ \bar{q} = y - tu;\quad t \text{ is a parameter.}$$

This gives a linear isomorphism $T^*V = V \times V$ where (p,q) are coordinates in the first and $(\bar{p},\bar{q})$ in the second factor. Let $\Omega_t = dp \wedge dq - d\bar{p} \wedge d\bar{q}$; then $\Omega_t = 2t\Omega$.

Given a function $f(x,y)$ on V, the Taylor expansion of the function $f(x-tv, y+tu)/2t$ in powers of tu and tv is just the series $I_t(f)$. In the new coordinates this equals $f(p,q)/2t$.

To prove that I_t is a Poisson Lie algebra homomorphism one needs to show that

$$\{I_t(f), I_t(g)\}_\Omega = I_t(\{f,g\}_\omega).$$

Indeed,

$$\{I_t(f), I_t(g)\}_\Omega = \Big\{\frac{1}{2t}f(p,q), \frac{1}{2t}g(p,q)\Big\}_\Omega = 2t\Big\{\frac{1}{2t}f(p,q), \frac{1}{2t}g(p,q)\Big\}_{\Omega_t} =$$

$$\frac{1}{2t}\{f(p,q), g(p,q)\}_{\Omega_t} = \frac{1}{2t}\{f(p,q), g(p,q)\}_{dp\wedge dq} = I_t(\{f,g\}_\omega)$$

(the second equality due to the fact that $\Omega_t = 2t\Omega$).

Likewise,

$$J_{t^2}(f) = \frac{1}{2t}(f(p,q) - f(\bar{p},\bar{q})),$$

so J_t is a homomorphism for a similar reason.

Next we indicate how to construct homomorphisms I_t and J_t on an arbitrary symplectic manifold (M,ω); this construction is less canonical than the one in linear space.

Consider the symplectic manifold $(M \times M, \omega \ominus \omega)$, and let π be the projection on the first factor. The diagonal $M \subset M \times M$ is a Lagrangian submanifold. According to Weinstein's theorem, a neighbourhood U of the diagonal is symplectomorphic to a neighbourhood V of the zero section in T^*M; we identify U and V. Given $f \in C^\infty(M)$, consider $f\ \pi \in C^\infty(U) = C^\infty(V)$. Compose this function on V with the fiberwise dilation in T^*M with the coefficient t, divide by t, and denote the resulting function on V by $I_t(f)$.

The map $I_t : C^\infty(M) \to C^\infty(V)$ is a Poisson Lie algebra homomorphism. Restricting to the infinite order formal neighbourhood of the zero section in T^*M, one obtains a Poisson Lie algebra homomorphism $I_t : C^\infty(M) \to Sect\ S^*(TM)$. The construction of J_t is similar, and the proof that both maps are homomorphisms repeats that in the linear case.

Finally, we prove Theorem 1.8. As before, we work in Darboux coordinates (x, y) and the dual basis (u, v). Introduce a convenient formalism.

Let

$$\partial_x \wedge \partial_y = \partial_x \otimes \partial_y - \partial_y \otimes \partial_x,\ \partial_u \wedge \partial_v = \partial_u \otimes \partial_v - \partial_v \otimes \partial_u.$$

The formula for the $\star$-product from 1.8 can be written as

$$f \star_t g = Tr\ e^{t(\partial_x \wedge \partial_y)}(f \otimes g),$$

where $Tr\ (f \otimes g) = fg$. Similarly, one has the next result.

Lemma. *The pairing*

$$< , >: (Sect\ S^k(TV))^{\otimes 2} \to C^\infty(V)$$

from 1.8 is given by the formula

$$< f, g >= Tr\ e^{(\partial_u \wedge \partial_v)}(f \otimes g).$$

Proof. The value

$$\frac{(\partial_u \otimes \partial_v - \partial_v \otimes \partial_u)^m}{m!}\ (u^I v^J \otimes u^K v^L)$$

equals $(-1)^{|J|} I! J!$ if $K = J, L = I, m = |I| + |J|$, and zero otherwise.

Recall also that the homomorphism I_t was written in 1.3 as

$$I_t(f) = \frac{1}{2t} e^{t(u\partial_y - v\partial_x)}(f).$$

Thus

$$< I_s(f), I_t(g) >= \frac{1}{4st} Tr\ e^{(\partial_u \wedge \partial_v)}\big(e^{s(u\partial_y - v\partial_x)} \otimes e^{t(u\partial_y - v\partial_x)}\big)(f \otimes g) =$$

$$\frac{1}{4st} Tr\ e^{(\partial_u \wedge \partial_v)} e^{st(u\partial_y - v\partial_x)\otimes(u\partial_y - v\partial_x)}(f\otimes g) = \frac{1}{4st} Tr\ e^{st(\partial_x\wedge\partial_y)}(f\otimes g) = \frac{1}{4st}\ f\star_{st} g.$$

This is the first identity of Theorem 1.8.

In view of the formulas

$$\{f,g\}_{t^2} = \frac{1}{2t}\ (f\star_t g - g\star_t f), \quad J_{t^2} = I_t + I_{-t},$$

the second identity of Theorem 1.8 is equivalent to

$$< I_s(f) + I_{-s}(f), I_t(g) + I_{-t}(g) >= 2 < I_s(f), I_t(g) > -2 < I_t(g), I_s(f) > \qquad (****)$$

Split $I_t(f)$ into the series $I_t^{ev}(f)$ and $I_t^{odd}(f)$, even and odd in t, respectively. Then the left hand side of (****) equals $4 < I_s^{ev}(f), I_t^{ev}(g) >$. The right hand side equals

$$2(< I_s^{ev}(f), I_t^{ev}(g) > + < I_s^{odd}(f), I_t^{odd}(g) > - < I_t^{ev}(g), I_s^{ev}(f) > - < I_t^{odd}(g), I_s^{odd}(f) >).$$

Note that the pairing $< , >$ is symmetric on even degree tensors and skew-symmetric on odd degree ones. Note also that I_t^{ev} is an odd degree tensor, and I_t^{odd} is an even degree one. Combining these two remarks it follows that the right hand side of (****) also equals $4 < I_s^{ev}(f), I_t^{ev}(g) >$. Theorem 1.8 is proved.

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