NOTES

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The Four-Vertex Theorem Revisited—Two Variations on the Old Theme

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The classical four-vertex theorem states that a closed imbedded smooth plane curve has at least four vertices; a vertex is an extremum of curvature. There are many proofs of this theorem—see, e.g. [**B-G, O**] and the references therein. In some recent works the four-vertex theorem was considered and generalized from the viewpoint of symplectic topology and Sturm theory—see [**A1-4, T, G-M-O, O-T**]. We present here two results inspired by the four-vertex theorem.

With a smooth plane curve γ another curve Γ, called its caustic (or evolute) is associated: Γ is the envelope of the family of normal lines to γ. Generically the curvature extrema of γ are simple maxima or minima (so that the second derivative of curvature does not vanish at its critical points). Assume that γ is in general position in this sense. Then Γ is a smooth front, that is a singular curve such that at each point the tangent line is well defined—see Fig. 1 (in more technical terms, a front is the projection of a smooth Legendrian curve in the contact manifold of contact elements of the plane to this plane; the general position condition means that the Legendrian curve has only simple tangency with the fibers of the projection—see [**A2**] for details). Singularities of Γ correspond to vertices of γ; generically they are cusps shown in the figure. The four-vertex theorem states that the caustic of a closed imbedded curve has at least four cusps.

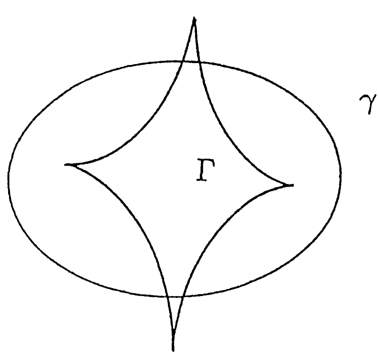

Figure 1

Parametrize the curve γ by length parameter; we write $\gamma(t)$ and use primes to indicate the derivative with respect to t. Then the normal line to γ at point $\gamma(t)$ is generated by the acceleration vector $\gamma''(t)$. We formulate the four-vertex theorem once again: the envelope of the family of lines through points $\gamma(t)$ in the directions of $\gamma''(t)$ has at least four cusps. This is the statement we generalize in our first theorem.

Consider a smooth plane curve γ. Call its parametrization (not necessarily by length) *definite* if the acceleration vector revolves all the time in the same sense; analytically: $\gamma'''(t) \wedge \gamma''(t) \neq 0$ for all t, where $\wedge$ denotes the determinant of two vectors. Let Γ be the envelope of the family of lines $l(t)$ through points $\gamma(t)$ in the direction of $\gamma''(t)$. As before Γ is a front; the points of γ corresponding to its cusps will be referred to as (generalized) vertices.

Theorem 1. *A generic convex closed smooth curve with a definite parametrization has at least four generalized vertices.*

Proof: To start with we claim that through every point x in the plane at least one (actually two) of the lines $l(t)$ passes. Indeed, consider the function $f(t) = (\gamma(t) - x) \wedge \gamma'(t)$. This function on the circle has a maximum at some point t_0. Then $0 = f'(t_0) = (\gamma(t_0) - x) \wedge \gamma''(t_0)$, that is the vectors $\gamma(t_0) - x$ and $\gamma''(t_0)$ are collinear. Thus $x \in l(t_0)$. Since the lines $l(t)$ are tangent to Γ we conclude that from each point in the plane there exists a tangent line to Γ.

Next consider the front Γ. It is oriented by the vectors $\gamma''(t)$. Since the parametrization of γ is definite, the tangent direction to Γ revolves in the same sense all the time, and its total turn is 2π. That is, the Gauss map of Γ is one-to-one. If Γ has no cusps then it is a closed convex curve, and there are no tangent lines to Γ from points inside it. This contradicts the previous paragraph.

Alternatively, a somewhat messy computation, which we omit, shows that vertices of γ are critical points of the function $g(t) = \gamma'''(t) \wedge \gamma''(t)/(\gamma''(t) \wedge \gamma'(t))^2$ (if t is the arc length parameter then $g(t)$ is the negative of the curvature). This function on the circle has at least one maximum and one minimum, hence γ has at least two vertices.

Finally we want to show that $g(t)$ has at least two local maxima and two local minima. Suppose not; then Γ has only two cusps. Consider a locally constant function $\phi(x)$ in the complement of Γ whose value at point x equals the number of tangent lines to Γ through x. The value of this function increases by 2 as x crosses Γ from the locally concave to the locally convex side—see Fig. 2. Let x be sufficiently far away from Γ. Since the Gauss map is one-to-one, $\phi(x) = 2$ (indeed there exist exactly two tangent lines to Γ from every point of the circle at infinity; by continuity the same holds for sufficiently distant points x).

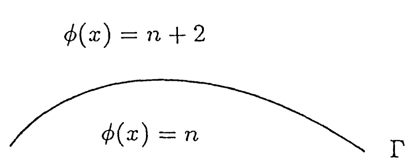

Figure 2

Consider the line through two cusps of Γ (which well may coincide); assume it is horizontal—see Fig. 3. Then the height function restricted to Γ attains either minimum or maximum (or both) not in a cusp. Assume it is maximum; draw the horizontal line l through it. Since Γ lies below this line, $\phi = 2$ above it. Therefore

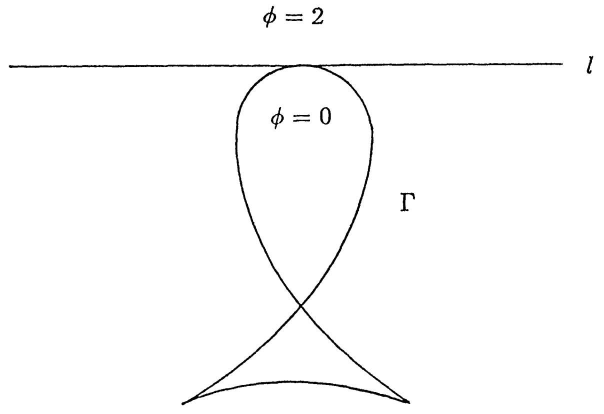

Figure 3

$\phi(x) = 0$ immediately below l, and there are no tangent lines to Γ from x. This contradicts the first paragraph of the proof. Q.E.D.

We pose some questions. First, is the assumption that the parametrization is definite really needed? If it fails, the front Γ will have branches that go to infinity.

Secondly, can one show that the function $g(t)$ has "many" critical points analytically? An interesting example is the affine parametrization characterized by $\gamma''(t) \wedge \gamma'(t) = 1$ for all t (it is not necessarily definite). In this case there exist at least six affine vertices—see **[G-M-O]** for a modern treatment.

Thirdly, call a generalized diameter of $\gamma(t)$ a chord which is collinear with the acceleration vectors $\gamma''(t)$ at both end points. Said differently, a generalized diameter is a double tangent line of Γ. Does γ always have at least 2 diameters? It is the case for the length parametrization.

Now we proceed to the second theorem. Given a closed plane curve γ we are interested in the following *tripod* configurations: three perpendiculars to γ dropped from one point that make angles of $2\pi/3$—see Fig. 4.

Theorem 2. *For any smooth convex closed curve there exist at least two tripod configurations.*

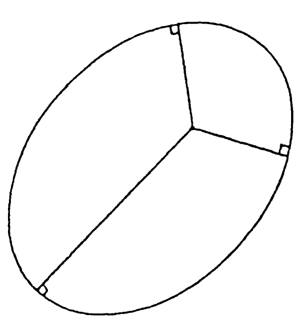

Figure 4

Proof: Let $\gamma(t)$ be a length parametrization, and let $\alpha(t)$ be the angle made by $\gamma'(t)$ with a fixed direction. Because of convexity we may (and will) use α as a parameter on γ. The curvature of γ is $\alpha'(t) = |\gamma''(t)|$.

Choose an origin O inside γ and let $p(t)$ be the signed distance from O to the line $l(t)$, i.e. $p = \gamma \wedge \gamma''/|\gamma''|$. Let $q = \gamma \wedge \gamma'$; we consider it as a function of α. One has:

$$\frac{dq}{d\alpha} = \frac{dq}{dt}\frac{dt}{d\alpha} = \frac{\gamma \wedge \gamma''}{|\gamma''|} = p.$$

Now consider the function $q(\alpha - 2\pi/3) + q(\alpha) + q(\alpha + 2\pi/3)$. It has a minimum and a maximum on the circle, say at points α_1 and α_2. Since p is the derivative of q we have:

$$p(\alpha_i - 2\pi/3) + p(\alpha_i) + p(\alpha_i + 2\pi/3) = 0, \quad i = 1, 2.$$

Consider three lines $l(\alpha - 2\pi/3)$, $l(\alpha)$ and $l(\alpha + 2\pi/3)$. They make an equilateral triangle; let a be the length of its side and A its area. Then

$$a(p(\alpha - 2\pi/3) + p(\alpha) + p(\alpha + 2\pi/3)) = 2A$$

—see Fig. 5 (it does not matter whether the origin lies inside or outside the triangle). For α_1, α_2 the left-hand side vanishes and the triangle degenerates to a point. We obtain two tripods. Q.E.D.

The reader may remember a version of the tripod theorem from his school years: there exists a point inside a triangle, whose angles are less than $2\pi/3$, from which all sides are seen at angles $2\pi/3$. This point minimizes the sum of distances to the vertices.

We conclude with another question: can the convexity assumption in the tripod theorem be relaxed? Does it hold for self-intersecting curves?

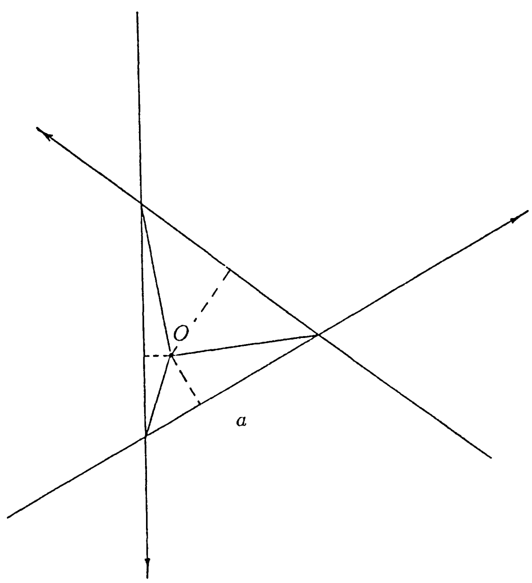

Figure 5

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Entire Functions Which Vanish at Infinity

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In a recent article in this journal David Armitage **[1]** showed how the pole-shoving technique used in 1885 by Carl Runge, and in most textbook accounts today of Runge's approximation theorem, can be exploited to prove a non-uniqueness theorem for the Radon transform. The idea is to push poles out to infinity through the region S between two confocal parabolas, while keeping the functions small outside S. Since every straight line spends only a compact amount of time in S, the limit entire function g vanishes at infinity along every line, although it is not identically 0. This is quite startling in view of the fact (maximum modulus principle) that the vanishing condition $\lim_{r \to +\infty} \sup_{|z|=r} |F(z)| = 0$ forces an entire function F to be 0. If one chooses a "transcendental swath" $S := \{x + iy : x \geq 1, ae^x \leq y \leq be^x\}$ $(0 < a < b)$, then the non-zero entire function g produced even vanishes at infinity along every unbounded algebraic curve, since every such curve must escape from this S. This clever idea goes back to Harold Bohr **[2]**.