The (un)equal tangents problem

Serge Tabachnikov*

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1 The problem

Given a point A outside of a closed strictly convex plane curve γ, there are two tangent segments from A to γ, the left and the right ones, looking from point A.

Problem: *Does there exist a curve γ such that one can walk around it so that, at all moments, the right tangent segment is smaller than the left one?*

In other words, does there exist a pair of simple closed curves, γ and Γ, the former strictly convex, the latter containing the former in its interior, such that for every point A of Γ the right tangent segment to γ is smaller than the left one?

Over the years, I have polled numerous colleagues, mostly as a dinner table topic. Most of them thought that the answer was negative, and quite a few tried to provide a proof, but each attempt had a flaw. I invite the reader to think about this question too before reading any further.

Up until recently, I have believed that for any oval[1] γ and every closed curve Γ going around γ, there existed a point $A \in \Gamma$ from which the tangent segments to γ were equal. In fact, I conjectured in [7] that there existed at least *four* such points.

Figure 1 illustrates the situation for an ellipse. The extensions of the axes partition the plane into four quadrants marked + and − according to the sign of the difference between the left and the right tangent segments. The axes themselves are the locus of points from which the tangent segments

*Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA; e-mail: tabachni@math.psu.edu

[1]a smooth strictly convex closed curve.

are equal (call this set the *equitangent locus*). Any curve surrounding the ellipse intersects the axes at least four times.

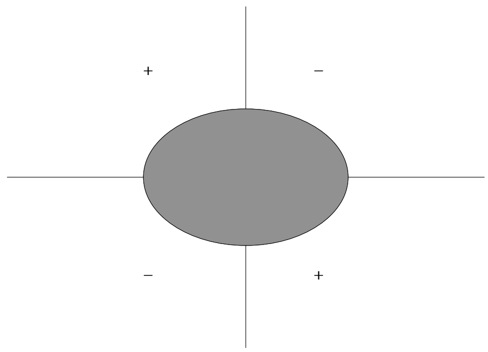

Figure 1: An ellipse and its equitangent locus

A certain confirmation to the above mentioned conjecture is the following, lesser known and quite beautiful, theorem [5], motivated by the flotation theory. Let γ be an oval and φ be an angle between 0 and π. Let Γ_φ be the locus of points in the exterior of γ from which γ is seen under angle φ. Then Γ_φ contains at least four points from which the tangent segments to γ are equal.

In the limit $\varphi \to \pi$, the points of Γ_φ from which the tangent segments to γ are equal correspond to the curvature extrema of γ. This implies the famous 4-vertex theorem: a plane oval has at last four distinct curvature extrema. In the limit $\varphi \to 0$, the points of Γ_φ from which the tangent segments to γ are equal correspond to the diameters of γ, that is, its binormal chords, and every diameter contributes two points on the curve "at infinity" Γ_0. This implies a well known fact that every oval has at least two diameters, the maximal one and a minimax one, corresponding to its least width.

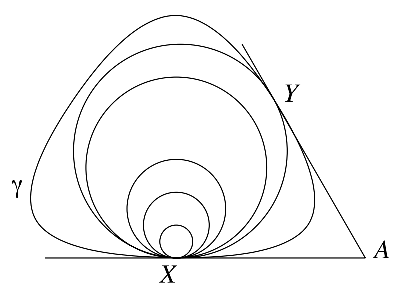

Figure 2: Bitangent circle

Here is another piece of evidence for the conjecture: *every tangent line to an oval γ contains at least two points from which the tangent segments to γ are equal.* Indeed, two tangent segments to an oval γ are equal if and only if there is a circle touching γ at these tangency points. Let ℓ be a tangent line to γ at point X. Consider the largest circle, tangent to ℓ at point X and contained in γ. This circle touches γ at another point, as needed, see Figure 2. Similarly, one considers the smallest circle, tangent to ℓ at point X and containing γ.

And yet, the surprising answer to the above formulated problem is YES! In this note, we construct such a curve.

2 Construction

One can reformulate our problem as follows. Let AX and AY be the tangent segments to an oval γ, and let α and β be the angles between XY and γ, see Figure 3. The question is whether points X and Y can make a complete circuit along γ so that always

$$\beta < \alpha \quad \text{and} \quad \alpha + \beta < \pi. \tag{1}$$

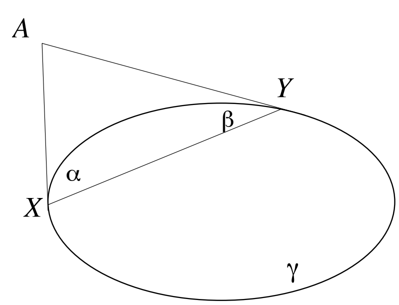

Figure 3: A chord of an oval

First we construct a convex polygon γ with a desired one-parameter family of chords, and then we approximate this polygon by an oval. Dealing with a convex polygon, we need to explain what we mean by "tangent lines" at its vertices. These are the support lines, that is, the lines thought a vertex that do not intersect the interior of the polygon.

The polygon γ is shown in Figure 4. It is a dodecagon, constructed by attaching congruent triangles, such as $A_1B_1A_2$, to the sides of a regular

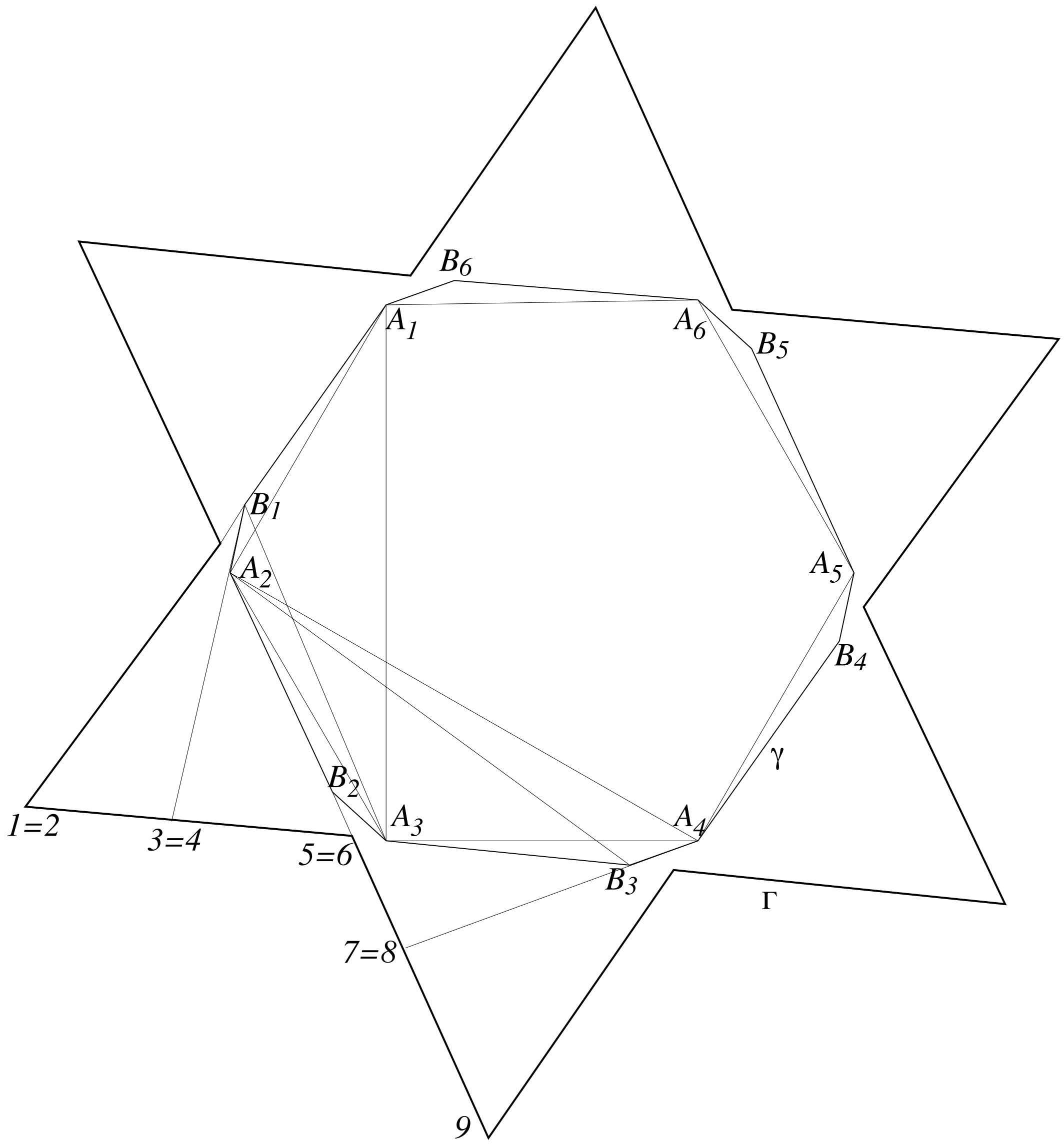

Figure 4: The dodecagon γ

hexagon $A_1 \dots A_6$. The point B_i is sufficiently close to the side A_iA_{i+1} and closer to A_{i+1} than to A_i. The polygon has the 6-fold rotational symmetry.

Now we describe the motion of the chord inside γ. When dealing with a smooth curve, a chord uniquely determines the tangent directions at its end-points. Since γ is a polygon, we describe a position of a chord by a triple whose first element is the chord, the second element is a support line at the first end-point, and the third element a support line at the second end-point.

The motion of the chord will be piecewise linear: in each step, either one end-point of the chord moves along a side of the polygon γ, or the end-points remain fixed but one of the support lines at the end-points revolves about this point. Thus, at each step, only one element of a triple changes.

Here is the whole process consisting of nine steps:

$$\begin{gathered}(A_1A_3, A_1B_1, A_3B_3) \to (B_1A_3, A_1B_1, A_3B_3) \to (B_1A_3, B_1A_2, A_3B_3) \to \\ (A_2A_3, B_1A_2, A_3B_3) \to (A_2A_3, A_2B_2, A_3B_3) \to (A_2B_3, A_2B_2, A_3B_3) \to \\ (A_2B_3, A_2B_2, B_3A_4) \to (A_2A_4, A_2B_2, B_3A_4) \to (A_2A_4, A_2B_2, A_4B_4). \end{gathered} \tag{2}$$

After that, the process repeats using the 6-fold rotational symmetry.

The 6-pronged star Γ around the dodecagon γ in Figure 4 is the locus of the intersection points of the tangent lines at the end-points of the moving chord inside γ, i.e., the locus of points labelled A in Figure 3. The points corresponding to the nine steps of the motion (2) are marked 1 to 9. We note that these points coincide pair-wise. Note that all the angles of Γ are equal to 60°.

We now check that the inequality $\beta < \alpha$, as in (1), holds during the motion (2). Since all the angles change monotonically, it suffices to check the inequalities at the first eight steps of (2). Let

$$\angle A_2A_1B_1 = \varphi,\ \angle A_1A_2B_1 = \psi,\ \angle A_2A_3B_1 = \theta,\ \angle B_3A_2A_4 = \delta.$$

We may make all these angles sufficiently small. By construction, $\varphi < \psi$. We also claim that

$$\varphi < \theta, \quad \varphi < \delta. \tag{3}$$

Indeed, since the inscribed angles subtended by the same arc of a circle are equal, one has

$$\theta = \angle A_2A_3C = \angle A_2A_1C > \angle A_2A_1B_1 = \varphi,$$

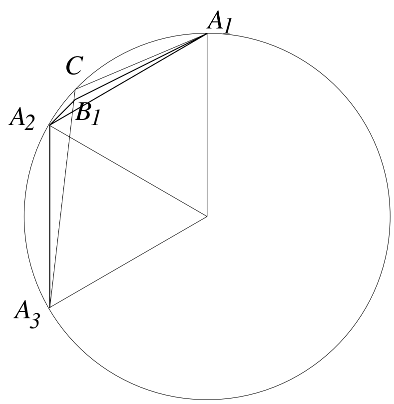

Figure 5: An angle inequality

see Figure 5, and the other inequality in (3) is similar.

The pairs of angles (β, α) for the first eight positions in (2) are easily found by elementary geometry:

$$(30^\circ + \varphi, 90^\circ - \varphi),\ (60^\circ + \varphi - \theta, 60^\circ - \varphi + \theta),\ (60^\circ - \theta - \psi, 60^\circ + \theta - \varphi),$$
$$(60^\circ - \psi, 60^\circ - \varphi),\ (\varphi, 60^\circ - \varphi),\ (30^\circ - \delta + \varphi, 30 + \delta - \varphi),$$
$$(30^\circ - \delta + \varphi, 30^\circ + \psi + \delta),\ (30^\circ + \varphi, 30^\circ + \psi).$$

For each pair, the inequality $\beta < \alpha$ holds, due to the inequalities (3) and $\varphi < \psi$, and the fact that φ and ψ are small enough. This completes our analysis of the polygonal example.

Next, we approximate the dodecagon γ by a smooth strictly convex curve, say, γ_1. If the approximation is fine enough, we obtain a one-parameter family of chords of γ_1, such as depicted in Figure 3, for which the inequality $\beta < \alpha$ still holds. Thus γ_1 is a desired example.

To be concrete, one may construct a C^1-smooth example by replacing each vertex of the dodecagon γ by an arc of a circle of a very small radius, and every side of γ by an arc of a circle of a very large radius. We assume that the small and the large circles share tangent directions at their common points. The resulting piecewise circular curve γ_1 is an example; see [1] concerning the fascinating geometry of piecewise circular curves.

We note that when γ is approximated by, say, a piecewise circular curve, the polygonal curve Γ also changes slightly. In particular, the pairs of coinciding points in Figure 4, such as 1 and 2, separate and become pairs of distinct close points. (One of the referees pointed out that approximation by a piecewise circular curve has the somewhat unwelcome feature that the equitangent locus contains entire regions adjacent to the arcs involved. The perturbed curve Γ will be disjoint from these regions.)

3 Remarks and comments

1. Given an oval γ, its equitangent locus is, generically, a curve. It may have component of the following types: starting and ending on γ (at its vertices); starting and ending at infinity; closed components;[2] and components starting on γ and ending at infinity. It is this last type of components that cannot be avoided by loops going around γ. The conjecture in [7] was that there existed at least four such components for every plane oval. See the computer-generated Figure 6 (courtesy of P. Giblin).

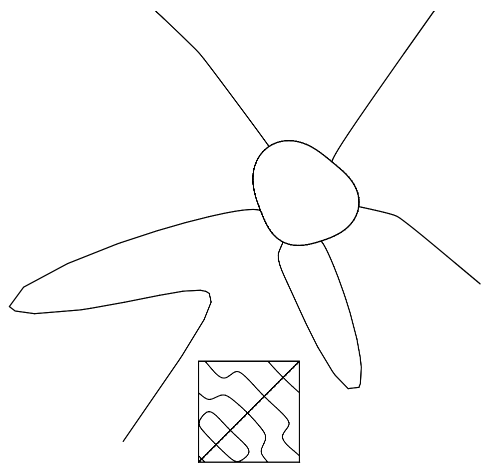

Figure 6: The equitangent locus and the respective cycle on the torus

2. As we mentioned, two tangent segments to an oval γ are equal if and only if there is a circle touching γ at these tangency points. The locus of centers of such bitangent circles is called the symmetry set of γ [2]. Thus our problem is closely related to the geometry of symmetry sets. Symmetry sets have attracted much interest, in particular, due to their applications to image recognition and computer vision, see [3, 6].

In particular, a binormal chord automatically gives a bitangent circle with the centre coinciding with the midpoint of the chord, and so gives a point at infinity on the equitangent locus. The corresponding branches of

[2]A closed component of the equitangent locus may contain γ in its interior.

the equitangent locus for the curve γ constructed in this article will start at infinity and go back to infinity without touching the curve γ (nor Γ, for that matter).

3. Generically, the set of pairs of tangency points (X, Y) of equal tangent segments AX and AY to an oval γ is a 1-cycle on the torus $\gamma \times \gamma$, symmetric with respect to the diagonal. The components of this cycle that are isotopic to the anti-diagonal are called *essential loops* in [4]. It was conjectured in [4] that there existed at least two essential loops; this conjecture is equivalent to the conjecture in [7], mentioned above.

A path around the oval γ is represented by a curve on the torus that is isotopic to the diagonal. Such a curve intersects each essential loop at least twice. The example constructed in this note is free from essential loops. In 2004, P. Giblin and V. Zakalyukin constructed an example of a non-convex plane curve free from essential loops, see Figure 7; this example was adapted from an earlier example of Zakalyukin [8] devised for a different purpose.

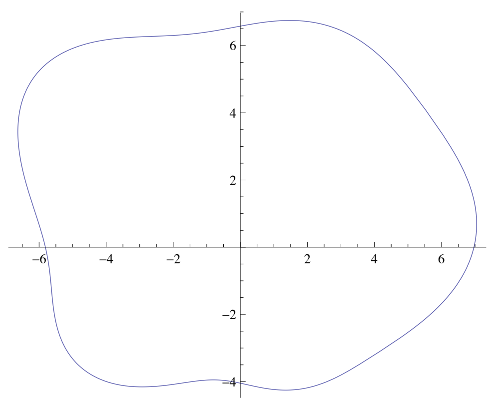

Figure 7: The example of Giblin and Zakalyukin (provided by P. Giblin)

4. If γ is a convex polygon, a definition of a tangent segment to γ from an exterior point, say, A, is needed. If A does not lie on the extension of a side, this is a support segment to the respective vertex of the polygon. If A belongs to an extension of a side, say, XY, then every segment AZ, with point Z on the side XY, counts as a tangent segment. Using this definition, one can define the equitangent locus Δ for a convex polygon γ; this is also a polygonal curve. Note that, for a polygon, the relation between exterior points and chords, as depicted in Figure 3, is not one-to-one anymore.

See Figure 8 for Δ when γ is an obtuse triangle XYZ. Δ is made of segments of extensions of the sides and median perpendiculars of the triangle; it has four components that start on γ and go to infinity, and one component that starts and ends on the triangle. Similarly to Figure 1, we mark by $+$ and $-$ the components of the complement of the equitangent locus Δ.

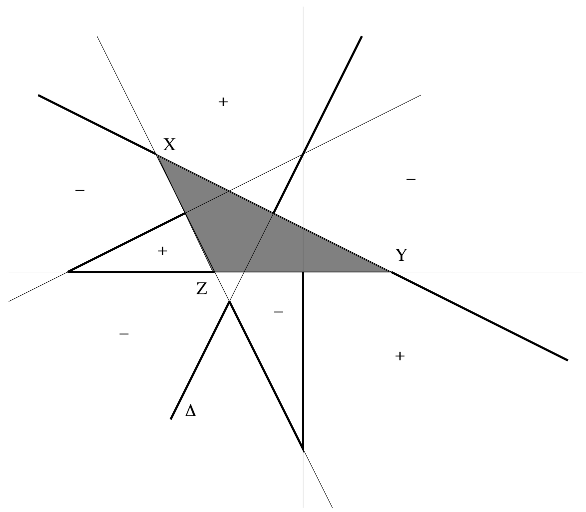

Figure 8: The equitangent locus Δ for triangle XYZ

5. Figure 9 depicts a similar picture for a regular hexagon. The set marked $+$ consists of eighteen regions, and the closure of this set is connected. There exists a curve surrounding the hexagon consisting of points where the difference of the left and right tangent segments is non-negative, intersecting the zero locus at its vertices.

The dodecagon in Figure 4 is a perturbation of the regular hexagon obtained by inserting a vertex in each edge in a symmetrical fashion and pushing out a small amount. This provides a resolution of the "positive" region in Figure 9, making it possible to find a surrounding curve that stays in the region where the length difference function is positive.

It appears that one can make a similar construction based on regular n-gons with $n \geq 5$.

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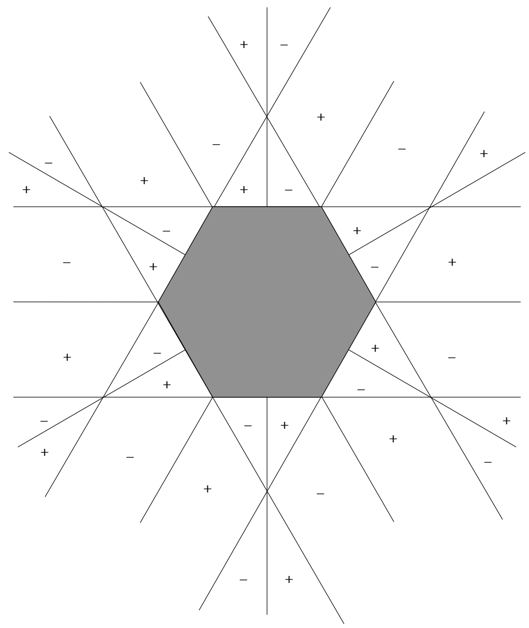

Figure 9: The equitangent locus for a regular hexagon

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