Existence and non-existence of skew branes

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Abstract

A skew brane is a codimension 2 submanifold in affine space such that the tangent spaces at any two distinct points are not parallel. We show that if an oriented closed manifold has a non-zero Euler characteristic χ then it is not a skew brane; generically, the number of oppositely oriented pairs of parallel tangent spaces is not less than $\chi^2/4$. We give a version of this result for immersed surfaces in dimension 4. We construct examples of skew spheres of arbitrary odd dimensions, generalizing the construction of skew loops in 3-dimensional space due to Ghomi and Solomon. We conclude with two conjectures that are theorems in 1-dimensional case.

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Key words: skew brane, skew loop, embedding, immersion

To S. Smale with admiration

1 Introduction

In mid-1960s, H. Steinhaus conjectured that every closed smooth curve in 3-dimensional space has parallel tangent lines. Shortly after that, B. Segre [14, 15] constructed examples of curves without parallel tangent lines but showed that no such curve can lie on the sphere (and therefore, due to the affine invariance of the problem, on an ellipsoid). Call a curve without parallel tangents a *skew loop*. Geometrical and topological study of skew loops and their multi-dimensional analogs has become an active research topic.

Let us briefly survey known results. The supply of skew loops is abundant: for example, every knot type in $\mathbf{R}^3$ can be realized by a skew loop [25], and so can every homotopy class of loops in a flat 3-torus [17]. The aversion of skew loops to ellipsoids was extended to convex quadratic surfaces in [6] and to non-convex ones in [22], see also [5, 16] for C^1 (as opposed to C^2) versions of these results. Ghomi and Solomon [6] proved a converse statement: if a convex surface is not quadratic then it carries a skew loop. See [18, 19] for generalizations of the Blaschke theorem that convex surfaces whose plane sections are centrally-symmetric must be quadratic. Another non-existence result is proved in [22]: no skew loops lie on a ruled developable disc.

A multi-dimensional version of a skew loop is called a *skew brane*.[1] A skew brane $f : M^n \to \mathbf{R}^{n+2}$ is an immersion such that the tangent spaces $df(T_xM)$ and $df(T_yM)$ are not parallel for all $x \neq y$. Pairs of parallel tangent spaces correspond to self-intersections of the image of the tangent Gauss map $M \to G_n(n+2)$ in the Grassmann manifold of n-dimensional subspaces in $\mathbf{R}^{n+2}$. Since dim $G_n(n+2) = 2n$, these self-intersections generically occur in isolated points and cannot be destroyed by small perturbations of f. If M is oriented, one distinguishes the cases when parallel tangent spaces have the same or the opposite orientations; we refer to them as positively and negatively parallel tangent spaces.

The aversion of skew loops to quadric surfaces extends to skew branes: no skew brane can lie on a quadratic hypersurface of any signature [22, 16]; for spheres, this was proved in [24]. The paper by Lai [10] has been overlooked in the more recent literature on the subject. Apparently unaware of the work of Segre, Lai provides an example of a skew loop in $\mathbf{R}^3$, and of a torus $T^2 \subset \mathbf{R}^4$ which is free from pairs of negatively parallel tangent planes. The main result of [10] is the following theorem on non-existence of skew branes.

Theorem 1 *Let $M^{2n} \subset \mathbf{R}^{2n+2}$ be a closed oriented embedded submanifold with non-zero Euler characteristic χ. Then there exists a pair of distinct points $x, y \in M$ such that the tangent spaces T_xM and T_yM are negatively parallel. For a generic submanifold M, the number of such (unordered) pairs (x, y) is not less than $\chi^2/4$.*

By a generic submanifold we mean the one whose Gauss map is an immersion with transversal self-intersections; see Section 2 for a justification.

The papers by Lai [9, 10] continue the work by Blaschke [2] and Chern and Spanier [3] concerning embedded surfaces in $\mathbf{R}^4$. In this case, we add to Theorem 1 the following result. Let $M \subset \mathbf{R}^4$ be a closed oriented immersed surface of genus g, generic in the sense that its Gauss map is an immersion with transversal self-intersections and that self-intersections of M in $\mathbf{R}^4$ are transversal. Then each double point is assigned a sign; denote by d the algebraic number of double points.

Theorem 2 *M has at least $|d^2 - (1-g)^2|$ pairs of negatively parallel tangent planes.*

In Section 3 we construct examples of skew branes; this is the main result of the present paper. Let $\mathbf{R}^{2n}$ be a subspace in $\mathbf{R}^{2n+1}$ and $S^{2n-1} \subset \mathbf{R}^{2n}$ the unit sphere. We consider this sphere as a codimension 2 submanifold in $2n+1$-dimensional space.

[1]The terms "skew loop" and "skew brane" were coined by M. Ghomi and B. Solomon; see [5, 7, 20, 21] for recent study of other classes of non-degenerate embeddings of manifolds into affine and projective spaces.

Theorem 3 *There exists a small perturbation of S^{2n-1} in $\mathbf{R}^{2n+1}$ that makes it into a skew brane.*

For $n = 1$, a much stronger result is proved in [6]: given a non-centrally symmetric smooth closed curve γ in the horizontal plane, there exists a skew loop on the cylinder over γ that projects diffeomorphically on γ ("cylinder lemma"). We believe that an analog of the cylinder lemma holds for odd-dimensional spheres; see Conjecture 4.1 and a more general Conjecture 4.2 in Section 4.

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2 Topological obstructions to the existence of skew branes

In this section we prove Theorem 1 and Theorem 2. The proof of Theorem 1 in [10] is based on a detailed analysis of the Schubert cell decomposition of Grassmann manifolds made in [9]; our proof uses characteristic classes of vector bundles and is somewhat more streamlined.

Consider the tangent Gauss map $F : M^{2n} \to G^+_{2n}(2n+2)$ to the Grassmann manifold of oriented $2n$-dimensional subspaces in $\mathbf{R}^{2n+2}$. Let σ be the involution of the Grassmannian that inverts the orientation of every $2n$-dimensional subspace. The proof consist of computing the homology class $F_*[M]$ in the group $H_{2n}(G^+_{2n}(2n+2), \mathbf{Z})$ where $[M]$ is the fundamental class of M, and of the homology intersection number $F_*[M] \cap \sigma_* F_*[M]$. We suppress the coefficients from the notation of homology and cohomology.

Denote by ξ^{2n} and ν^2 the tautological vector bundles: the former has the oriented $2n$-plane E, and the latter its orthogonal complement, as the fiber over E, considered as a point in $G^+_{2n}(2n+2)$. The bundles are oriented and so is the Grassmannian; the tangent bundle is as follows: $TG^+_{2n}(2n+2) = \mathrm{Hom}(\xi, \nu)$ (this formula provides the orientation of $TG^+_{2n}(2n+2)$). Denote the fundamental class by $c \in H_{4n}(G^+_{2n}(2n+2))$, and let $x = e(\xi) \in H^{2n}(G^+_{2n}(2n+2))$ and $y = e(\nu) \in H^2(G^+_{2n}(2n+2))$ be the Euler classes. Denote by dot the pairing between homology and cohomology classes.

The cohomology ring of the Grassmannian $G^+_{2n}(2n+2)$ is well known. We need the following result formulated below without proof (cf. [9]).

Lemma 2.1 *One has:* $c \cdot x^q = 2,\ c \cdot y^p = 2(-1)^n,\ xy = 0.$

Denote by $u \in H_{2n}(G^+_{2n}(2n+2))$ and $v \in H_{4n-2}(G^+_{2n}(2n+2))$ the dual homological Euler classes. Set $w = v \cap \ldots \cap v \in H_{2n}(G^+_{2n}(2n+2))$. It follows from Lemma 2.1 that

$$u \cap u = 2, w \cap w = 2(-1)^n, u \cap w = 0. \tag{1}$$

As a consequence, u and w are linearly independent.

It is known that $H_{2n}(G_{2n}^+(2n+2)) = \mathbf{Z}^2$, see, e.g., [9]. It follows that one may take u and w for a basis in $H_{2n}(G_{2n}^+(2n+2); \mathbf{Q})$.

Denote by η the normal bundle of M in $\mathbf{R}^{2n+2}$, and let $r = [M] \cdot e(\eta)^n$.

Lemma 2.2 *One has:* $F_*[M] = (\chi/2)u + (-1)^n (r/2) w$.

Proof. Note that the induced bundles $F^*(\xi)$ and $F^*(\nu)$ are the tangent and the normal bundles of M. One has:

$$F_*([M]) \cap u = F_*([M]) \cdot x = [M] \cdot F^*(x) = [M] \cdot F^*(e(\xi)) = [M] \cdot e(TM) = \chi.$$

Likewise,

$$F_*([M]) \cap w = F_*([M]) \cdot y^n = [M] \cdot F^*(y^n) = [M] \cdot F^*(e(\xi)^n) = [M] \cdot e(\eta)^n = r.$$

Write $F_*[M] = au + bw$, then (1) implies that $a = \chi/2$ and $b = (-1)^n (r/2)$. □

It is a classical fact that the Euler class of the normal bundle of a manifold, embedded in Euclidean space, vanishes. That is, $r = 0$. Using the fact that $\sigma_*(u) = -u$, one concludes that

$$\sigma_* F_*[M] \cap F_*[M] = -\left(\frac{\chi}{2}\right) u \cap \left(\frac{\chi}{2}\right) u = -\frac{\chi^2}{2}.$$

This implies the first statement of Theorem 1. If the Gauss image $F(M^{2n}) \subset G_{2n}^+(2n+2)$ is an immersed submanifold which intersects $\sigma(F(M^{2n}))$ transversally then their homological intersection equals the algebraic number of the intersection points. In particular, there are no fewer intersection points than the absolute value of the homological intersection number, as needed.

To prove Theorem 2, we use another classical fact: for a generic immersion of an oriented closed $2n$-dimensional manifold in $\mathbf{R}^{4n}$ with normal bundle η, the number $-[M] \cdot e(\eta)$ is an algebraic count of double points of M. More specifically, in our situation M is a surface, and the following lemma holds.

Lemma 2.3 *One has:* $[M] \cdot e(\eta) = -2d$.

Proof. Since $\mathbf{R}^4$ is contractible, $[M] \cap [M] = 0$. On the other hand, this homological self-intersection can be computed as follows. Choose a generic section γ of the normal bundle η and let M_ε be the result of pushing M slightly along this section. Then every double point of M contributes two points (with the same sign) to $M \cap M_\varepsilon$, and each zero of γ contributes one point to this intersection. It follows that $e(\eta) + 2d = 0$. □

It follows that $F_*([M]) = (1-g)u + dw$. Using $\sigma_*(w) = -w$ and (1), one concludes that

$$\sigma_* F_*[M] \cap F_*[M] = 2(d^2 - (1-g)^2).$$

This implies Theorem 2.

Now we justify the notion of generic immersion. For this, we need the following lemma in which "generic" is understood as belonging to an open and dense subset in the space of smooth maps with an appropriate topology; see, e.g., [8] or [23].

Lemma 2.4 *For a generic immersion $M^n \to \mathbf{R}^{n+2}$, the tangent Gauss map $G : M \to G_n(n+2)$ is an immersion with transverse self-intersections.*

Proof. Locally, M is represented as the graph of a smooth map $U^n \to \mathbf{R}^2$ where U is a domain in $\mathbf{R}^n$. Thus one has two functions, say, u and v, of variables $x = (x_1, \dots, x_n)$. The tangent space to M at the point corresponding to $x \in U$ is parallel to the graph of the linear map $\mathbf{R}^n \to \mathbf{R}^2$ with the $n \times 2$ matrix $A(x) = (u_{x_i}, v_{x_i}),\ i = 1, \dots, n$. Hence, in an appropriate local chart of the Grassmannian, the Gauss map $G : U \to G_n(n+2)$ is given by $x \mapsto A(x)$.

The Gauss map is an immersion if the $2n \times n$ matrix $B = (u_{x_i x_j}, v_{x_i x_j})$, $i, j = 1, \dots, n$, has full rank n. This matrix is formed by two symmetric $n \times n$ matrices of second partial derivatives. Consider the $n(n+1)$-dimensional space of $2n \times n$ matrices formed by two symmetric $n \times n$ matrices, and let Δ be its algebraic subvariety that consists of matrices of rank $n-1$ or less.

Claim: codim $\Delta \geq n+1$.

Assuming this claim, one proceeds as follows. In the space of 2-jets of smooth maps $U^n \to \mathbf{R}^2$, consider the algebraic subvariety Σ of the maps for which the matrix B is not of full rank. By the above claim, this subvariety has codimension $n+1$. Then the Thom transversality theorem (see, e.g., [8, 23]) implies that the 2-jet extension of a generic map $U^n \to \mathbf{R}^2$ avoids Σ, and we conclude that the Gauss map is generically an immersion.

Let us prove the claim. The following argument was suggested by the referee; it is an improvement of the original author's one.

A $2n \times n$ matrix $C = (S, T)$ where S and T are symmetric $n \times n$ matrices has rank $< n$ if and only if $K := \ker S \cap \ker T \neq 0$. Let X_k be the space of matrices C with $\dim K = k$. Then the projection $C \to K$ provides a fibering of X_k over the Grassmannian $G_k(n)$; the fiber over a space K is open in the space of pairs of self-adjoint maps $K^\perp \to K^\perp$. It follows that

$$\dim X_k = k(n-k) + 2((n-k)(n-k+1)/2) = n(n+1) - k(n+1) \geq n(n+1) - (n+1),$$

as claimed.

It remains to prove that the Gauss map generically has transverse self-intersections. The argument is similar to the above, so we sketch it briefly.

Let $(u(x), v(x))$ and $(\bar{u}(x), \bar{v}(x))$ be two pairs of functions of $x = (x_1, \dots, x_n)$ parameterizing open neighborhoods in M whose Gauss images in $G_n(n+2)$ intersect. The intersection condition is that $u_{x_i} = \bar{u}_{x_i},\ v_{x_i} = \bar{v}_{x_i}$ for $i = 1, \dots, n$, and the intersection is not transverse if the $2n \times 2n$ matrix of second partial derivatives, made of the $n \times n$ blocks $u_{x_i x_j}, v_{x_i x_j}, \bar{u}_{x_i x_j}, \bar{v}_{x_i x_j}$, is degenerate. All combined, this gives $2n+1$ relations on the first and second derivatives. The relevant space here consists of quadruples of n-vectors (representing gradients of $u, \bar{u}, v, \bar{v}$) and quadruples of symmetric $n \times n$ matrices (representing their Hessians), and in this space one has a codimension $2n+1$ subvariety, corresponding to the above described relations. This corresponds to an algebraic subvariety $\bar{\Sigma}$ in the space of pairs of 2-jets of smooth maps $U^n \to \mathbf{R}^2$ with codim $\bar{\Sigma} \geq 2n+1$. Then the multi-jet transversality theorem implies that the 2-jet extension of a pair of generic maps $U^n \to \mathbf{R}^2$ avoids $\bar{\Sigma}$, and hence the generic self-intersections of the Gauss image are transverse. □

Remarks 2.1 1. In the case $n = 1$, the Grassmannian is the product of spheres: $G_2^+(4) = S^2 \times S^2$. Denote the two factors by S_1 and S_2. The main result of [2, 3] is that $F_*[M] = (\chi(M)/2)([S_1] + [S_2])$. The relation with classes u and v is as follows: $u = [S_1] + [S_2], v = [S_2] - [S_1]$. The map d that takes a plane to its orthogonal complement is the identity on S_1 and the antipodal involution on S_2. Under the identification $\mathbf{R}^4 = \mathbf{C}^2$, the space of complex lines in $\mathbf{C}^2$ identifies with $\{N, S\} \times S_2$ where N and S are the poles of S_1.

2. Suppose that an immersed surface M in the statement of Theorem 2 is totally real, that is, the tangent plane to M is never a complex line. Then $F_*[M] = a[S_2]$ for some $a \in \mathbf{Z}$, and the algebraic number of negatively parallel tangent planes is zero. Alternatively, the normal bundle of a totally real immersed surface is isomorphic to the tangent bundle, which implies that $d^2 = (1-g)^2$. Likewise, this number is zero for an immersed surface M with non-vanishing normal curvature, see [12]. Indeed, it is proved in [12] that the normal Euler class of M is $\pm 2\chi(M)$, that is, $d = \pm\chi(M)$.

3. It follows from the above proof that χ is even. Indeed, a theorem of Seifert states: a closed orientable embedded manifold $M^{2n} \subset \mathbf{R}^{2n+2} = \mathbf{C}^{n+1}$ has an even Euler characteristic. It is proved in [9] that $\chi(M)$ is twice the algebraic number of positively oriented complex points of M.

4. The Grassmannian $G_{2n}^+(2n+2) = G_2^+(2n+2)$ is the space of oriented great circles in the unit sphere S^{2n+1}. The space of oriented geodesics of a Riemannian manifold has a canonical symplectic structure (assuming it is a smooth manifold), see, e.g., [1]. Thus $G_{2n}^+(2n+2)$ is a symplectic manifold, and $\omega^{2n} \neq 0$ where ω is the symplectic form. In the previous notation, the cohomology class of ω is y.

3 Odd-dimensional skew sphere

Now we shall construct an odd-dimensional skew sphere of Theorem 3.

Let $M^{m-1} \subset \mathbf{R}^m$ be a smooth strictly convex closed hypersurface containing the origin. Denote by $h : S^{m-1} \to \mathbf{R}$ the support function of M, that is, $h(x)$ is the distance from the origin to the tangent hyperplane to M for which the unit vector $x \in S^{m-1}$ is the outward normal. It will be convenient to extend functions defined on the sphere to homogeneous functions of degree 1 on $\mathbf{R}^m$. Extending h in this way, one has, by the Euler formula:

$$xh_x = h, \quad xh_{xx} = 0. \tag{2}$$

Here and elsewhere we use the following vector notation: for m-dimensional vectors x, y and a function g of m variables, one has:

$$xy = \sum_{i=1}^{m} x_i y_i, \quad xdy = \sum_{i=1}^{m} x_i dy_i, \quad g_x = (g_{x_1}, \ldots, g_{x_m}),$$

$$xg_{xx} = (\sum_{i=1}^{m} x_i g_{x_i x_1}, \ldots, \sum_{i=1}^{m} x_i g_{x_i x_m}),$$

etc.

The hypersurface M is parameterized by the unit sphere as follows (this result is well known in convex geometry).

Lemma 3.1 *Let $y(x) \in M$ be the point at which the outward unit normal to M is $x \in S^{m-1}$. Then $y = h_x$.*

Proof. Consider the hypersurface M' given by the formula $y(x) = h_x$. To prove that the tangent hyperplane to M' at y is orthogonal to x, one needs to check that the 1-form xdy vanishes on M'. Indeed, $xy = h$ by (2), and hence

$$xdy = d(xy) - ydx = dh - h_x dx = 0.$$

The distance to the tangent hyperplane with the normal vector x is $xy = xh_x$, which equals h according to (2). Thus h is the support function of M' and therefore $M' = M$. □

It follows from (2) that, at point $x \in S^{m-1}$, the linear operator h_{xx} annihilates the normal direction x and preserves the tangent space $T_x S^{m-1}$. The convexity of M implies that, on the tangent space, h_{xx} is non-degenerate (for example, h_{xx} is the identity if $h = 1$ and hence M is the unit sphere). Denote the restriction of h_{xx}^{-1} on $T_x S^{m-1}$ by $A(x)$ and extend A to the normal direction, spanned by x, as the zero map.

We will construct a skew brane in $\mathbf{R}^{m+1}$ as a section of the vertical cylinder over $M \subset \mathbf{R}^m \subset \mathbf{R}^{m+1}$. More specifically, let $f : S^{m-1} \to \mathbf{R}$ be a smooth function. Define

$$N = \{(h_x(x), f(x)) |\ x \in S^{m-1}\} \subset \mathbf{R}^m \times \mathbf{R} = \mathbf{R}^{m+1},$$

and let $\phi : S^{m-1} \to N$ be the parameterization map.

Let us find out when the tangent spaces to N are parallel. As before, assume that f is extended to $\mathbf{R}^m$ as a homogeneous function of degree 1.

Lemma 3.2 *The tangent spaces to N are parallel at distinct points $\phi(x_1)$ and $\phi(x_2)$ if and only if $x_2 = -x_1$ and $A(f_x)(x_2) = A(f_x)(x_1)$.*

Proof. Let us describe the normal 2-plane to N at point $\phi(x)$. This plane is generated by the vector $(x, 0)$ and a vector $(\xi, 1)$ where $\xi \in T_x S^{m-1}$. We claim that $\xi = -A(f_x)$.

Indeed, let $v \in T_x S^{m-1}$ be a test vector, $v = \sum_j v_j \partial_{x_j}$. Then

$$d\phi(v) = \sum_j (v_j h_{x_i x_j}, v_j f_{x_j}), \quad i = 1, ..., m.$$

It follows that $d\phi(v)$ is orthogonal to $(\xi, 1)$ if and only if $v h_{xx} \xi + v f_x = 0$, or $v(h_{xx}\xi + f_x) = 0$. Since v is an arbitrary tangent vector to S^{m-1}, the projection of the vector $h_{xx}\xi + f_x$ to $T_x S^{m-1}$ is zero. This implies the claim.

Finally, the span of vectors $(\xi(x_2), 1)$ and $(x_2, 0)$ coincides with that of $(\xi(x_1), 1)$ and $(x_1, 0)$ if and only if $x_2 = -x_1$ and $\xi(x_2) = \xi(x_1) + t x_1$. Since $\xi(x_2)$ and $\xi(x_1)$ are orthogonal to x_1, one has $\xi(x_2) = \xi(x_1)$. $\Box$

Thus N is a skew brane if and only if

$$A(-x)(f_x(-x)) \neq A(x)(f_x(x)) \tag{3}$$

for all $x \in S^{m-1}$. Decompose the functions h and f into the even and odd components with respect to the antipodal involution of the sphere $x \mapsto -x$:

$$h = h_{ev} + h_{odd}, \quad f = f_{ev} + f_{odd}.$$

Then one has a similar decompositions of the operator A and the gradient vector field f_x. Note that

$$(f_x)_{ev} = (f_{odd})_x, \ (f_x)_{odd} = (f_{ev})_x.$$

Decomposing (3) into even and odd parts yields:

$$A_{ev}((f_{ev})_x) + A_{odd}((f_{odd})_x) \neq 0 \tag{4}$$

for all $x \in S^{m-1}$.

If M is sufficiently close to the sphere then h is close to 1 and A is close to the identity. Thus we may assume that A_{ev} is invertible and rewrite (4) as

$$(f_{ev})_x + B((f_{odd})_x) \neq 0 \tag{5}$$

where $B(x) = A_{ev}^{-1} A_{odd}$ is an odd field of linear maps of the tangent spaces $T_x S^{m-1}$.

Note that (5) cannot hold for all x if h, and therefore A, is even: then $B = 0$ and the function f_{ev} must have critical points on the sphere. Note also that (5) cannot hold if $m - 1$ is even since there exist no non-vanishing vector fields on even-dimensional spheres. From now on, assume that $m = 2n \geq 4$.

Lemma 3.3 *Let M^m be a connected manifold and B a smooth field of linear automorphisms of the cotangent spaces to M satisfying the following property: for every smooth function $f : M \to \mathbf{R}$ there exists a smooth function $g : M \to \mathbf{R}$ such that $B(df) = dg$. Then $B = c \,\mathrm{Id}$ for some constant c.*

Proof. Being a local statement, we may assume that M is a contractible domain in $\mathbf{R}^m$. In local coordinates $x_1, \ldots, x_m$, the linear map B is given by a matrix $b_{ij}(x)$. The condition is that, for every f, the 1-form $\sum_{ij} b_{ij} f_{x_j} dx_i$ is closed.

Choose $j \in \{1, \ldots, m\}$ and let $f(x) = x_j$. Then the 1-form $\sum_i b_{ij}(x) dx_i$ is closed, and hence there exists a function $g^j(x)$ such that $b_{ij} = g^j_{x_i}$. The 1-form $\sum_{ij} g^j_{x_i} f_{x_j} dx_i = \sum_j f_{x_j} dg^j$ is closed for every f, therefore so is $\sum_j g^j d(f_{x_j})$.

Again fix $j \in \{1, \ldots, m\}$ and let $f(x) = x_j^2/2$. Then the 1-form $g^j dx_j$ is closed, hence g^j depends only on x_j. Finally, fix $j, k \in \{1, \ldots, m\}$, $j \neq k$, and let $f(x) = x_j x_k$. Then the 1-form $g^j(x_j) dx_k + g^k(x_k) dx_j$ is closed, and therefore $g^j_{x_j} = g^k_{x_k} = c$. It follows that $b_{ij} = c\delta_{ij}$, as claimed. □

One has the following corollary.

Corollary 3.1 *If M is not centrally symmetric then there exists a function f_{odd} on S^{m-1} such that $B((f_{odd})_x)$ is not a gradient vector field.*

Proof. A parallel translation of the origin changes the support function h by addition of a linear function and does not affect h_{xx}. The hypersurface M is centrally symmetric if and only if, after a parallel translation, h is even, that is, A is even. Since M is not centrally symmetric, $A_{odd} \neq 0$ and $B \neq 0$.

Identify the tangent and cotangent spaces of the sphere by the Euclidean structure. Let U be a small domain on S^{m-1}. Then every function $f : U \to \mathbf{R}$ can be extended to S^{m-1} as an odd function. If $B(f_x)$ is a gradient for every such f then, by the preceding lemma, $B = c \,\mathrm{Id}$ in U. This equality then must hold everywhere on the sphere. But B is odd, hence $B = 0$, a contradiction. □

Our plan of proof of Theorem 3 is to construct functions h and f on S^{2n-1} so that (5) holds everywhere on S^{2n-1}. We set $h = 1 + \varepsilon g$ where g is an odd function and ε is a small parameter. Then $h_{xx} = Id + \varepsilon g_{xx} + O(\varepsilon^2)$ and $A = Id - \varepsilon g_{xx} + O(\varepsilon^2)$. It follows that $B = -\varepsilon g_{xx} + O(\varepsilon^2)$, and (5) can be rewritten as

$$(f_{ev})_x \neq \varepsilon g_{xx}((f_{odd})_x) + O(\varepsilon^2). \tag{6}$$

We wish to construct a function f and an odd function g so that, for all sufficiently small ε and a certain constant $c > 0$, depending on f and g, one has

$$|(f_{ev})_x - \varepsilon g_{xx}((f_{odd})_x)| > c\varepsilon \tag{7}$$

everywhere on the sphere. This will imply (6), and hence, for sufficiently small ε, the desired inequality (5) as well.

Let $z = (z_1, \ldots, z_n)$ be coordinates in $\mathbf{C}^n$. Choose distinct non-zero reals $a_1, \ldots, a_n$, and let $f_{ev} = \sum_i a_i |z_i|^2$. Denote by ξ the unit Hopf vector field on

S^{2n-1} whose value at point z is the vector iz. The orbits of ξ are circles which we call Hopf circles. Thus f_{ev} is an even, ξ-invariant Morse-Bott function on S^{2n-1} with exactly n critical Hopf circles $C_1, \ldots, C_n$.

Lemma 3.4 *There exist odd smooth functions f_{odd} and g on S^{2n-1} such that $g_{xx}((f_{odd})_x) \cdot \xi = 2$ on each critical circle $C_i, i = 1, \ldots, n$.*

Assuming this lemma, denote the vector field $g_{xx}((f_{odd})_x)$ by v. Let c be a constant such that $|v(x)| < c$ everywhere on the sphere (this constant exists by compactness of the sphere). Given $\varepsilon > 0$, let U_ε be a ξ-invariant neighborhood of the critical set $C_1 \cup \ldots \cup C_n$ such that $|(f_{ev})_x| > 2\varepsilon c$ outside of U_ε. We claim that (7) holds outside of U_ε. Indeed, $|(f_{ev})_x - \varepsilon v| > 2\varepsilon c - \varepsilon c = \varepsilon c$, as claimed.

Next, consider the situation inside U_ε. Since $v \cdot \xi = 2$ on the critical circles, we can choose ε so small that $v \cdot \xi > 1$ inside U_ε. Note that $(f_{ev})_x$ is orthogonal to ξ. Therefore $|((f_{ev})_x - \varepsilon v) \cdot \xi| = \varepsilon v \cdot \xi > \varepsilon$, and hence $|((f_{ev})_x - \varepsilon v)| > \varepsilon$ inside U_ε. In particular, (7) holds for this choice of ε.

Thus, Theorem 3 will follow, once we prove Lemma 3.4.

Proof of Lemma 3.4. We construct the desired functions in a neighborhood of each critical circle, symmetric with respect to the antipodal involution, and then extend them to the sphere.

Let C be one such Hopf circle. Consider $\mathbf{C}^2 \subset \mathbf{C}^n$ such that C is a Hopf circle therein. Choose coordinates (z_1, z_2) in $\mathbf{C}^2$ so that $z_i = x_i + \sqrt{-1} y_i,\ i = 1, 2$, and C is given by $x_2 = y_2 = 0$. The Hopf field is given by the formula:

$$\xi = -y_1 \partial x_1 + x_1 \partial y_1 - y_2 \partial x_2 + x_2 \partial y_2.$$

The functions $f = f_{odd}$ and g will depend only on (x_1, y_1, x_2, y_2).

It is straightforward to compute the operator g_{xx}, the gradient f_x and the dot product $g_{xx}(f_x) \cdot \xi$. Introduce the differential operator depending on f:

$$D = f_{x_1} \partial x_1 + f_{y_1} \partial y_1 + f_{x_2} \partial x_2 + f_{y_2} \partial y_2.$$

Then,

$$g_{xx}(f_x) \cdot \xi = x_1 D(g_{y_1}) - y_1 D(g_{x_1}) + x_2 D(g_{y_2}) - y_2 D(g_{x_2}). \tag{8}$$

Let us look for f and g in the following form:

$$f = a(x_1, y_1) + x_2 b(x_1, y_1) + y_2 d(x_1, y_1),\ g = u(x_1, y_1) + x_2 v(x_1, y_1) + y_2 w(x_1, y_1)$$

where a, u are odd and homogeneous of degree 1, b, d, v, w are even and homogeneous of degree 0. Switch to polar coordinates $x_1 = r \cos \alpha, y_1 = r \sin \alpha$. On the circle C, one has $x_2 = y_2 = 0, r = 1$, and α is a coordinate. Then (8) becomes

$$bv' + dw' + (u + u'')a' \tag{9}$$

where prime is $d/d\alpha$. Set:

$$a = u = 0, \quad b = -w = \cos 2\alpha, \quad d = v = \sin 2\alpha,$$

and the expression (9) gets identically equal to 2. In terms of the Cartesian coordinates,

$$f = x_2(x_1^2 - y_1^2) - 2y_2x_1y_1, \quad g = 2x_2x_1y_1 - y_2(x_1^2 - y_1^2)$$

on the unit sphere (of course, their homogeneous of degree 1 extension to $\mathbf{C}^2$ is given by dividing these formulas by $x_1^2 + x_2^2 + y_1^2 + y_2^2$, but this is not important to us).

It remains to extend the odd functions from a neighborhood of the union of the critical Hopf circles to the whole sphere as an odd function. Clearly, one can extend a smooth function φ from an open subset U to a function Φ on the whole sphere (for example, using a smooth cut-off function for the set U). If φ is odd then one may modify $\Phi(x)$ by replacing it with $(1/2)(\Phi(x) - \Phi(-x))$. The latter is an odd function on the sphere that coincides with φ on U. This completes the proof. □

4 Two conjectures

Generalizing [6], we make the following conjecture.

Conjecture 4.1 *For every non-centrally symmetric $M^{2n-1} \subset \mathbf{R}^{2n}$ there exists a skew brane $N^{2n-1} \subset \mathbf{R}^{2n+1} = \mathbf{R}^{2n} \times \mathbf{R}$ which is a section of the vertical cylinder over M.*

Conjecture 4.1 follows from a more general conjecture of independent interest.

Conjecture 4.2 *Let M be a closed manifold with zero Euler characteristic, α a non-closed differential 1-form on it. Then there exists a smooth function $f : M \to \mathbf{R}$ such that the 1-form $df + \alpha$ has no zeroes on M.*

Deducing Conjecture 4.1 from Conjecture 4.2. One may rephrase (5) as

$$d(f_{ev}) + \alpha \neq 0 \tag{10}$$

everywhere on S^{2n-1}; here α is the 1-form dual to the vector field $B((f_{odd})_x)$. By Corollary 3.1, we may choose $f_{odd} : S^{2n-1} \to \mathbf{R}$ so that α is not closed. Consider the quotient space $M = \mathbf{RP}^{2n-1} = S^{2n-1}/\mathbf{Z}_2$. Then α descends as a non-closed 1-form $\bar{\alpha}$ on M. Conjecture 4.2 implies that there exists a function $\bar{f} : \mathbf{RP}^{2n-1} \to \mathbf{R}$ such that $d\bar{f} + \bar{\alpha}$ is nowhere vanishing. Then $\bar{f}$ lifts to an even function on S^{2n-1} for which (10) holds. □

Remark 4.1 In dimension one, Conjecture 4.2 holds. Indeed, $\alpha = g(x)dx$ with $c := \int_0^1 g(x)dx \neq 0$; here x is the coordinate on the circle $\mathbf{R}/\mathbf{Z}$. Then $g - c$ is a derivative, $g(x) - c = -f'(x)$, and $df + \alpha = cdx$. It follows that, in dimension one, Conjecture 4.1 holds as well: this is precisely the Cylinder Lemma of [6].

We conclude with comments on Conjecture 4.2. If the 1-form is exact, i.e., $\alpha = dg$ for some function g, then zeros of $df + \alpha$ are the critical points of the function $f + g$. Morse theory implies that the number of these critical points is bounded below in terms of the topology of the manifold M, e.g., by the sum of its Betti numbers. Likewise, if α is a closed 1-form, one has similar lower bounds for the number of zeros of a 1-form in the cohomology class of α, this time, due to the Morse-Novikov theory. Conjecture 4.2 states that if neither Morse nor Morse-Novikov theory applies then there is no topological reason for the existence of zeros of the form $df + \alpha$.

Conjecture 4.2 also resembles the following theorem of Polterovich [13] and Laudenbach-Sikorav [11]. It is known in symplectic topology that, under certain assumptions, the number of intersection points of a Lagrangian submanifold in a symplectic manifold with its translate under a Hamiltonian isotropy is at least the sum of its Betti numbers. This result of symplectic topology generalizes Morse inequalities. The theorem of Polterovich and Laudenbach-Sikorav asserts that if L is a middle-dimensional submanifold of a symplectic manifold which is not Lagrangian and which has zero Euler characteristic then L can be displaced from itself by a Hamiltonian vector field. The proof of this theorem makes use of Gromov's h-principle (the technique of convex integration). I believe that a similar approach will be instrumental in a proof of Conjecture 4.2.

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