

On skew loops, skew branes and quadratic hypersurfaces

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Abstract

A skew brane is an immersed codimension 2 submanifold in affine space, free from pairs of parallel tangent spaces. Using Morse theory, we prove that a skew brane cannot lie on a quadratic hypersurface. We also prove that there are no skew loops on embedded ruled developable discs in 3-space.

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To V. I. Arnold on the occasion of the 65-th anniversary

1 Introduction and formulation of results

In a lecture, given in 1966, H. Steinhaus asked the question: Do there exist smooth closed space curves without a pair of parallel tangents? Call such a curve a *skew loop*. Steinhaus conjectured that skew loops did not exist. The problem was solved by B. Segre in [5, 6]. Segre constructed examples of skew loops in 3-space but proved that no skew loops lie on an ellipsoid or an elliptic paraboloid. Thus skew loops have an aversion to quadrics.

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These results were recently extended and refined by M. Ghomi and B. Solomon [2]. On the one hand, the following theorem holds: *Nonnegatively curved quadrics admit no skew loops.* In particular, no skew loops lie on one sheet of a 2-sheeted hyperboloid. On the other hand, the converse is true: *Any non-quadratic surface with a point of positive curvature admits a skew loop.*

Another recent result on this subject [10]: *Every knot type can be realized by a skew loop.*

A multi-dimensional version of the problem reads as follows. Let $f : M^n \rightarrow V^{n+2}$ be an immersion of a manifold to an affine space. One asks whether there are pairs of points $x, y \in M$ such that the tangent spaces $df(T_x M)$ and $df(T_y M)$ are parallel in V . Such pairs correspond to self-intersections of the image of the tangent Gauss map $M \rightarrow G_n(V)$ in the Grassman manifold $G_n(V)$ of n -dimensional non-oriented subspaces in V . Since $\dim G_n(V) = 2n$, these self-intersections generically occur in isolated points. An immersed closed connected codimension 2 manifold is called a *skew brane* if it is free from pairs of parallel tangents.

J. White [9] considered the case of M^n immersed into the unit sphere S^{n+1} in Euclidean space V^{n+2} . The main observation in this case is as follows: if xy is a double normal of M , that is, the segment xy is perpendicular to M at both end points, and the points x and y are not antipodal then the tangent spaces $T_x M$ and $T_y M$ are parallel in V . This made it possible to estimate below the number of parallel tangents of M by Morse theory of the distance squared function on $M \times M$, studied in a related paper [8]. See [3, 4] for recent results on double normals of immersed submanifolds in Euclidean space.

The goal of this note is to extend the above mentioned results on non-existence of skew loops and skew branes from convex quadratic surfaces to quadratic hypersurfaces of all signatures. Here is our main result.

Theorem 1 *Let Q^{n+1} be a non-degenerate quadratic hypersurface in a vector space V^{n+2} and $f : M^n \rightarrow Q$ be a generic smooth immersion of a closed manifold. Then there exist distinct points $x, y \in M$ such that the tangent spaces $df(T_x M)$ and $df(T_y M)$ are parallel in V .*

In particular, there are no generic skew loops on one-sheeted hyperboloids and hyperbolic paraboloids, a new result compared to [2, 5, 6]. The meaning of the general position assumption is explained in section 2.

We give applications to two types of quadratic surfaces in 3-space: to one sheet of the 2-sheeted hyperboloid $z^2 - x^2 - y^2 = 1, z > 0$ and to 1-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$.

Theorem 2 (i) *Let γ be a generic smooth closed immersed curve on one sheet of the 2-sheeted hyperboloid, and let a be the number of its double points. Then γ has at least $a+2$ pairs of tangent lines, parallel in the ambient space.*
(ii) *Let γ be a generic smooth closed immersed curve on the 1-sheeted hyperboloid, and assume that γ is space-like: the quadratic form $x^2 + y^2 - z^2$, evaluated on its tangent vector, is everywhere positive. Let a be the number of double points of γ and b the number of intersections with its antipodal image. Then γ has at least $a + b$ pairs of tangent lines, parallel in the ambient space. If $a = b = 0$ then there are at least 2 pairs of parallel tangents.*

Statement (i) of Theorem 2 holds for spherical curves that do not intersect their antipodal images – see [9].

Another result concerns non-existence of skew loops on developable surfaces in 3-dimensional Euclidean space. Let D^2 be a disc, embedded in the Euclidean space V^3 as a smooth ruled developable surface, and let γ be a generic smooth closed immersed curve in D .

Theorem 3 *There exists a pair of points on γ with tangent lines, parallel in the ambient space.*

Since cylinders are ruled developable surfaces, this theorem is in sharp contrast with a result from [2]: *the cylinder over any non centrally symmetric, strictly convex plane oval contains a skew loop*. This skew loop goes around the cylinder and is not contained in a simply connected domain. Moreover, it was pointed out by D. Burago that Theorem 3 fails if the the developable surface contains flat pieces; in this case it may not be ruled.

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2 Proof of Theorem 1

Assume that

$$Q^{n+1} = \{x \in V^{n+2} \mid x \cdot x = 1\}$$

where dot denotes a (pseudo)Euclidean structure of signature $(p+1, q)$, $p+q = n+1$, $p, q \geq 0$. The quadratic hypersurface Q is the unit (pseudo)sphere. Let (x_1, ξ_1) and (x_2, ξ_2) be two contact elements of Q ; here $x_i \in Q$ is the foot point and $\xi_i \subset T_{x_i}Q$ is a hyperplane in the tangent space to Q . Assume that $x_2 \neq \pm x_1$, and let U be the 2-dimensional subspace spanned by x_1 and x_2 .

Lemma 2.1 *The spaces ξ_1 and ξ_2 are parallel in V if and only if they are both perpendicular to U .*

Proof. Two n -dimensional spaces with the same orthogonal complement are parallel. Conversely, let ξ_1 and ξ_2 be parallel. Then their orthogonal complements in V coincide. Note that the position vector x_i is orthogonal to the (pseudo)sphere Q and, therefore, to ξ_i . Hence the orthogonal complement of ξ_i contains x_i , and the common orthogonal complement of ξ_1 and ξ_2 is U . \square

Remark 2.2 One can reformulate Lemma 2.1 as follows. Consider the 2-parameter family of parallel codimension 2 affine subspaces in V , orthogonal to a given 2-dimensional vector space U . Then the locus of tangency of these affine subspaces with Q is the intersection curve $U \cap Q$. This curve is a geodesic on Q for the (pseudo)metric, induced from V .

The next lemma extends a result of White [9] from spheres to all non-degenerate quadratic hypersurfaces. Let $f : M^n \rightarrow Q$ be an immersion. Consider the function on $M \times M$ given by the formula:

$$\phi(x, y) = f(x) \cdot f(y). \tag{1}$$

Lemma 2.3 *Let $f(y) \neq \pm f(x)$. The spaces $df(T_x M)$ and $df(T_y M)$ are parallel in V if and only if (x, y) is a critical point of the function ϕ .*

Proof. Let U be the the 2-dimensional space spanned by $f(x)$ and $f(y)$. A point (x, y) is critical for the function ϕ if and only if $f(x)$ is orthogonal to $df(T_y M)$ and $f(y)$ is orthogonal to $df(T_x M)$. Also $f(x)$ is orthogonal to $T_{f(x)}Q$ and hence to $df(T_x M)$, and likewise, $f(y)$ is orthogonal to $df(T_y M)$.

Thus (x, y) is critical for ϕ if and only if U is orthogonal to the spaces $df(T_x M)$ and $df(T_y M)$. By Lemma 2.1, this holds if and only if these two spaces are parallel. \square

Remark 2.4 A special degenerate case of a quadratic hypersurface is the graph of a quadratic form. Let $V = W \oplus \mathbf{R}$ where space W has a non-degenerate quadratic form τ , and let

$$Q = \{(x, z) \in V \mid z = \tau(x)\}.$$

Denote by π the projection of V to W . Let $f : M^n \rightarrow Q$ be an immersion and let $g = \pi \circ f$. Consider the function on $M \times M$ given by the formula:

$$\psi(x, y) = \tau(g(x) - g(y)).$$

Then an analog of Lemma 2.3 reads as follows: for $g(y) \neq g(x)$, the spaces $df(T_x M)$ and $df(T_y M)$ are parallel in V if and only if (x, y) is a critical point of the function ψ .

The proof of Theorem 1 proceeds as follows. Let $f : M^n \rightarrow Q^{n+1}$ be an immersed skew brane such that the self-intersection of M and its intersection with the antipodal image are transversal. This is the desired general position property. Consider the double and the antipodal double loci in $M \times M$:

$$A = \{(x, y) \in M \times M \mid y \neq x, f(y) = f(x)\},$$

and

$$B = \{(x, y) \in M \times M \mid f(y) = -f(x)\},$$

and let $D \subset M \times M$ be the diagonal. Then A and B are closed submanifolds of dimensions $n - 1$ (cf. [9]). Furthermore we require that ϕ in (1) is a Morse function on $M \times M - A \cup B \cup D$; this is also provided by the genericity of the immersion.

By Lemma 2.3, the function ϕ has no critical points on $M \times M$ off the sets A, B and D . We will show that this cannot be the case, and this contradiction will prove Theorem 1.

Recall that if a smooth function has a non-degenerate critical manifold then the tangent space to the critical manifold is the kernel of its Hessian. By Morse index one means the index of the Hessian in the normal bundle to the critical manifold – see, e.g., [1].

Lemma 2.5 *The manifolds D, A and B are critical manifolds of the function ϕ with the critical values $1, 1$ and -1 , respectively. The manifolds A and B are non-degenerate, and the respective Morse indices equal p and q . Let (x, x) be a point of the diagonal D , and assume that the (pseudo)Euclidean structure on $T_x M$, induced by f from V , is non-degenerate and has index r (which can be equal to either q or $q-1$). Then the Morse index of D at point (x, x) equals $n - r$.*

Proof. Consider a point $(x_1, x_2) \in M \times M$, and let (u_1, u_2) be a tangent vector to $M \times M$ at this point. Let $(\gamma_1(t), \gamma_2(t))$ be a curve on $M \times M$ such that $\gamma_i(0) = x_i$ and $\gamma'_i(0) = u_i$; $i = 1, 2$. Set: $q_i(t) = f(\gamma_i(t))$, and introduce the following notation: $q_i(0) = y_i, q'_i(0) = v_i$ and $q''_i(0) = \eta_i$. In particular, $f(x_i) = y_i$ and $df(u_i) = v_i$.

To compute the Hessian, we make calculations modulo t^3 . One has:

$$q_i(t) = y_i + tv_i + \frac{1}{2}t^2\eta_i,$$

and therefore

$$\phi(\gamma_1(t), \gamma_2(t)) = y_1 \cdot y_2 + t(v_1 \cdot y_2 + y_1 \cdot v_2) + \frac{t^2}{2}(\eta_1 \cdot y_2 + 2v_1 \cdot v_2 + y_1 \cdot \eta_2).$$

If $(x_1, x_2) \in D \cup A \cup B$ then $y_2 = \pm y_1$ (plus for $D \cup A$ and minus for B), and the linear terms vanish:

$$v_1 \cdot y_2 = y_1 \cdot v_2 = 0.$$

Hence D, A and B are critical manifolds. Since $q_i(t) \cdot q_i(t) = 1$ one has:

$$y_i \cdot \eta_i + v_i \cdot v_i = 0.$$

Therefore

$$\phi(\gamma_1(t), \gamma_2(t)) = 1 - \frac{t^2}{2}(v_1 - v_2) \cdot (v_1 - v_2) \tag{2}$$

for $(x_1, x_2) \in D \cup A$ and

$$\phi(\gamma_1(t), \gamma_2(t)) = -1 + \frac{t^2}{2}(v_1 + v_2) \cdot (v_1 + v_2) \tag{3}$$

for $(x_1, x_2) \in B$.

Assume that $(x_1, x_2) \in A$. According to (2), the Hessian of ϕ is the quadratic form on $T_{x_1}M \oplus T_{x_2}M$ which is the composition

$$T_{x_1}M \oplus T_{x_2}M \rightarrow T_{y_1}Q \rightarrow \mathbf{R},$$

where the first map is a linear epimorphism given by the formula

$$(u_1, u_2) \rightarrow df(u_1) - df(u_2), \quad (4)$$

and the second map is negative the quadratic form on Q given by the restriction of the (pseudo)Euclidean structure in V . The kernel of (4) is the tangent space to A at point (x_1, x_2) , and the Morse index is equal to the index of the restriction of the (pseudo)Euclidean structure to Q . This structure in V has signature $(p+1, q)$, and Q is given by the equation $x \cdot x = 1$. Therefore the restriction of the (pseudo)Euclidean structure to Q has signature (p, q) , and the Morse index is equal to p .

A similar argument, with (3) replacing (2), shows that the Morse index of the critical manifold B equals q .

An analogous argument works for the diagonal as well: by (2), the Hessian of ϕ is the composition

$$T_xM \oplus T_xM \rightarrow T_{f(x)}Q \rightarrow \mathbf{R},$$

where the first arrow (4) is a linear map onto $df(T_xM) \subset T_{f(x)}Q$. The restriction of the quadratic form from Q to $df(T_xM)$ has signature either $(p-1, q)$ or $(p, q-1)$, depending on the direction (“space-like” or “time-like”) of M at point $f(x)$. Thus r is either q or $q-1$. \square

We are ready to complete the proof of Theorem 1. Recall that we work toward a contradiction under the assumption that the function ϕ has no critical points outside the manifolds A, B and D . Consider two cases.

Case 1: $q > 0$. Our assumption implies that ϕ attains minimum -1 on B . On the other hand, by Lemma 2.5, the Morse index of B equals q , a positive number, hence B is not a minimal critical manifold.

Case 2: $q = 0, p = n + 1$. Then the quadratic hypersurface Q is the unit sphere, the cases studied in [9]. Note that, by Lemma 2.5, the diagonal D is a non-degenerate critical manifold with the Morse index n .

Recall the Morse inequalities for Morse-Bott functions, that is, functions with non-degenerate critical manifolds [1]. Let g be such a function on a

closed manifold L with critical submanifolds N_i , $i = 1, \dots, k$, whose Morse indices are equal to μ_i . Let P_t denote the Poincaré polynomial in variable t of a manifold with coefficients in a fixed field. Then the Morse inequalities can be written as follows:

$$\sum_{i=1}^k t^{\mu_i} P_t(N_i) = P_t(L) + (1+t)Q_t \quad (5)$$

where Q_t is a polynomial with non-negative coefficients.

Apply (5) in our situation. Let

$$P_t(A) = a_0 + a_1t + \dots + a_{n-1}t^{n-1}, P_t(B) = b_0 + b_1t + \dots + b_{n-1}t^{n-1}$$

and

$$P_t(D) = P_t(M) = 1 + d_1t + \dots + t^n.$$

Then the Morse inequalities read:

$$\begin{aligned} t^n(1 + d_1t + \dots + t^n) + t^{n+1}(a_0 + a_1t + \dots + a_{n-1}) + (b_0 + b_1t + \dots + b_{n-1}t^{n-1}) = \\ (1 + d_1t + \dots + t^n)^2 + (1+t)Q_t. \end{aligned}$$

The coefficient of t^n on the left hand side is 1, while on the right hand side it is not less than 2. This is a desired contradiction proving Theorem 1.

3 Loops on surfaces. Open questions

Let us prove Theorem 2. In the case of the hyperboloid $z^2 - x^2 - y^2 = 1$, one has: $p = 0, q = 2$. The set A consists of $2a$ isolated points (a self-intersection counts twice: as (x, y) and as (y, x)), each with the Morse index 0, and the set B is empty. The Morse index of the diagonal D equals 0. Let $2d_0, 2d_1$ and $2d_2$ be the number of critical points of the function ϕ outside $D \cup A$; these numbers are even since ϕ is invariant under the involution $(x, y) \rightarrow (y, x)$. The Morse inequalities read:

$$(1+t) + 2a + (2d_0 + 2d_1t + 2d_2t^2) = (1+t)^2 + (1+t)(q_0 + q_1t)$$

where $q_0, q_1 \geq 0$. It follows that

$$q_0 = 2a + 2d_0 \geq 2a, \quad 2d_1 = 1 + q_0 + q_1 \geq 1 + 2a, \quad 2d_2 \geq 1.$$

Hence $d_1 \geq 1 + a$, $d_2 \geq 1$, and $d_1 + d_2 \geq 2 + a$, as claimed.

Consider the hyperboloid $x^2 + y^2 - z^2 = 1$. In this case $p = q = 1$. The sets A and B consist of $2a$ and $2b$ isolated points, respectively, each with the Morse index 1. If γ is space-like then the Morse index of the diagonal D equals 0. The Morse inequalities read:

$$(1 + t) + (2a + 2b)t + (2d_0 + 2d_1t + 2d_2t^2) = (1 + t)^2 + (1 + t)(q_0 + q_1t)$$

where $q_0, q_1 \geq 0$. It follows that

$$2d_0 = q_0, \quad 2a + 2b + 2d_1 = 1 + q_0 + q_1, \quad 2d_2 = 1 + q_1.$$

If $a = b = 0$ then $d_1 \geq 1$, $d_2 \geq 1$, and $d_1 + d_2 \geq 2$. Also $2d_0 + 2d_2 = 1 + q_0 + q_1 \geq 2a + 2b$, and thus $d_0 + d_2 \geq a + b$.

Now we proceed to the proof of Theorem 3. Recall the classification of ruled developable surfaces (see, e.g., [7]). Such a surface is either a cylinder or a cone or consists of tangent lines to a space curve (called the edge of regression). In either case, the tangent space to the surface remains the same along each ruling.

Let D be a ruled embedded developable disc and γ a closed immersed curve in it. Then D is foliated by straight lines, the ruling of the developable surface. Unfold D to the plane to obtain a plane domain \bar{D} foliated by straight lines and an immersed curve $\bar{\gamma}$ in it. Assume that, for two distinct intersection points x, y of $\bar{\gamma}$ with a leaf \bar{l} of the foliation, the tangents $T_x\bar{\gamma}$ and $T_y\bar{\gamma}$ are parallel. Then the respective tangent lines to γ lie in the plane, tangent to D along the corresponding ruling l , and are parallel in this plane. Hence they are parallel in the ambient space. Thus Theorem 3 follows from the next lemma.

Lemma 3.1 *Let $\bar{\gamma}$ be a generic closed immersed curve in a simply connected plane domain, foliated by straight lines. Then there exist two distinct points $x, y \in \bar{\gamma}$ on the same leaf of the foliation, such that the tangents $T_x\bar{\gamma}$ and $T_y\bar{\gamma}$ are parallel.*

Proof. This proof was proposed by A. Malyutin; another proof was given by R. Matveyev.

Let \bar{l}_t , $t \in [0, 1]$, be the family of lines foliating the domain; we think of these lines as oriented from left to right. Assume that $\bar{\gamma}$ lies between \bar{l}_0 and \bar{l}_1 , and that these lines touch the curve – see Figure 1. Let $C_+(t)$ and

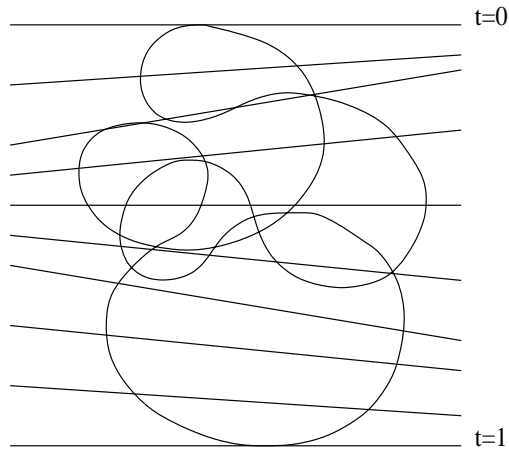


Figure 1: A curve in a domain foliated by lines

$C_-(t)$ be the rightmost and the leftmost points of $\bar{\gamma} \cap \bar{l}_t$. The curves $C_{\pm}(t)$ are piecewise smooth. Let $\alpha_{\pm}(t) \in [0, \pi]$ be the angle between $C_{\pm}(t)$ and \bar{l}_t .

The functions $\alpha_{\pm}(t)$ are not continuous but their discontinuities are easily described. The three types of discontinuities of $\alpha_-(t)$ are shown in Figure 2: in each case the graph of $\alpha_-(t)$ has a descending vertical segment. Likewise, there are three types of discontinuities of $\alpha_+(t)$, and in each case the graph has an ascending vertical segment.

Note that, for t close to 0, $\alpha_-(t)$ is close to 0, while $\alpha_+(t)$ is close to π . Similarly, for t close to 1, $\alpha_-(t)$ is close to π and $\alpha_+(t)$ is close to 0.

We claim that that the graphs of $\alpha_-(t)$ and $\alpha_+(t)$ have a common point not on a vertical segment of either graph. Approximate both functions by smooth ones so that the vertical segments of the graphs become very steep: $\alpha'_-(t) \ll 0$ and $\alpha'_+(t) \gg 0$ on the corresponding small intervals. Consider the function $\beta(t) = \alpha_+(t) - \alpha_-(t)$. For t near 0, $\beta(t) > 0$, and for t near 1, $\beta(t) < 0$. Let t_0 be the smallest zero of $\beta(t)$. Since β changes sign from positive to negative, $\beta'(t_0) \leq 0$. Therefore t_0 is not on the segments where $\alpha'_-(t) \ll 0$ or $\alpha'_+(t) \gg 0$. Thus t_0 is a desired common value of the two angles. \square

Remark 3.2 If the foliation consists of parallel lines, that is, the developable surface is (part of) a cylinder one can prove Lemma 3.1 as follows. Let xy be the longest segment of a leaf bounded by its intersection with the curve

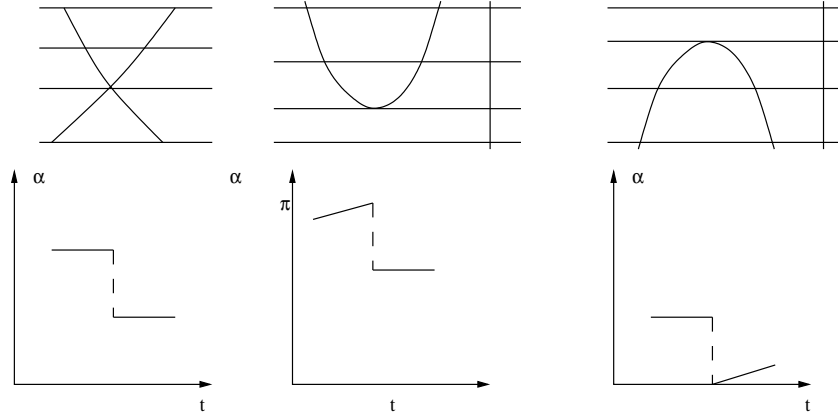


Figure 2: Discontinuities of the functions $\alpha_{\pm}(t)$

$\bar{\gamma}$. Then the tangents $T_x\bar{\gamma}$ and $T_y\bar{\gamma}$ are parallel.

Likewise, assume that the foliation consists of the lines passing through a point 0 , that is, the developable surface is (part of) a cone. Let xy be the segment of a leaf, $x, y \in \bar{\gamma}$, such that the ratio $|Ox|/|Oy|$ is greatest possible. Then the tangents $T_x\bar{\gamma}$ and $T_y\bar{\gamma}$ are parallel.

Remark 3.3 Let us describe an example of a simple skew loop on a developable disc, suggested by D. Burago. This developable surface contains flat regions and does not admit a continuous foliation by straight segments.

Take an equilateral triangle in the plane; its interior will be folded to make the desired embedding. Draw three lines near each corner, cutting across and not parallel to opposite sides. Fold smoothly along these lines so that the obtained surface consists of a flat hexagon with 3 triangular flaps (fins?) going more-or-less vertically. The curve γ is a simple smooth convex curve that goes along the perimeter of the original triangle. After the flaps are folded, the curve doesn't have parallel tangents in the ambient space.

We now pose some problems on skew loops and skew branes.

Problem 1. Give lower bounds on the number of pairs of parallel tangent spaces of M in Theorem 1 in terms of the topology of M . Such lower bounds on the number of double normals of a generic immersed submanifold in Euclidean space are found in [8, 3, 4] (the estimates by Pushkar' are sharp in some cases).

Problem 2. Extend Theorem 1 and lower bounds of Problem 2 to the

case when M is a wave front, Legendrian isotopic to a smooth manifold. We refer to [3, 4] for such results on double normals.

Problem 3. Our proof of Theorem 1 is based on Morse theory of the distance function $\phi : M \times M \rightarrow \mathbf{R}$. The proofs in [3, 4] make use of the function $\psi : M \times M \times S^N \rightarrow \mathbf{R}$ given by $\psi(x, y, v) = (x - y) \cdot v$; here M is immersed in Euclidean space \mathbf{R}^{N+1} . Critical points of ψ are double normals, and ψ is invariant under the free action of \mathbf{Z}_2 on the manifold $M \times M \times S^N$ sending (x, y, v) to $(y, x, -v)$. Can one apply this approach in the set-up of Theorem 1?

Problem 4. An immersion $f : M \rightarrow Q$ determines an immersion of M to the space of geodesics G of the pseudo-metric on Q : one assigns to a point of M its normal line. One is looking for self-intersections of the image of M in G . If the pseudo-metric on Q is a metric (then Q is either the sphere or the hyperbolic space) then the space of oriented geodesics \tilde{G} has a symplectic structure, and M is immersed to \tilde{G} as a Lagrangian submanifold. What can be said in the pseudo-metric case?

Problem 5. Construct examples of skew branes. For instance, let $N^n \subset W^{n+1}$ be a strictly convex *non centrally symmetric* closed hypersurface in vector space, and let n be odd. Can one construct a smooth function $g : N \rightarrow \mathbf{R}$ such that its graph in $V = W \times \mathbf{R}$ is a skew brane? For $n = 1$ this is the Cylinder Lemma of [2].

Problem 6. Do Theorems 1 and 3 hold without the general position assumption (conjecturally, they do)? The property of being a skew loop or a skew brane is not open: a small perturbation of a skew loop may create parallel tangents at a pair of close points $(x, y) \in M \times M$ near the diagonal $D \subset M \times M$. Thus one cannot simply argue by deforming a skew loop or a skew brane to a generic one.

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