

BIRKHOFF BILLIARDS ARE INSECURE

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ABSTRACT. We prove that every compact plane billiard, bounded by a smooth curve, is insecure: there exist pairs of points A, B such that no finite set of points can block all billiard trajectories from A to B .

1. Formulation of the result. Two points A and B of a Riemannian manifold M are called *secure* if there exists a finite set of points $S \subset M - \{A, B\}$ such that every geodesic connecting A and B passes through a point of S . One says that the set S blocks A from B . A manifold is called *secure* (or has the *finite blocking property*) if any pair of its points is secure. For example, every pair of non-antipodal points of the Euclidean sphere is secure, but a pair of antipodal points is not secure, so the sphere is insecure. A flat torus of any dimension is secure.

In the recent years, the notion of security has attracted a considerable attention, see [1, 2, 3, 4, 10, 11, 12, 14]. This notion extends naturally to Riemannian manifolds with boundary, in which case one considers billiard trajectories from A to B with the billiard reflection off the boundary.

In this note we consider a compact plane billiard domain M bounded by a smooth curve and prove that M is insecure. More specifically, one has the following local insecurity result. Consider a sufficiently short outward convex arc $\gamma \subset \partial M$ with end-points A and B (such an arc always exists).

Theorem 1.1. *The pair (A, B) is insecure.*

2. Proof of insecurity. Denote by T_n the polygonal line $A = P_0, P_1, \dots, P_{n-1}, P_n = B$, $P_i \in \gamma$, of minimal length; this is a billiard trajectory from A to B . If n is large then T_n lies in a small neighborhood of γ .

Working toward contradiction, assume that a finite set of points $S \subset M - \{A, B\}$ blocks every billiard trajectory from A to B . Decompose S as $S' \cup S''$ where the points of S' lie on the boundary and the points of S'' lie inside the billiard table. For n large enough, the trajectory T_n is disjoint from S'' . We want to show that there is a sufficiently large n such that the set $\mathbf{P}_n = \{P_1, \dots, P_{n-1}\}$ is disjoint from S' .

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Let s be the arc-length parameter and $k(s)$ the curvature of γ . Let σ be a new parameter on the arc γ such that $d\sigma = (1/2)k^{2/3}ds$. By rescaling the arc γ , we may assume that the range of σ is $[0, 1]$ with $\sigma(0) = A$ and $\sigma(1) = B$. Let $Q_0 = A, Q_1, \dots, Q_{n-1}, Q_n = B$ be the points that divide the σ -measure of γ into n equal parts, that is, $Q_m = \sigma(m/n)$. The next proposition describes the limiting distribution of the vertices of the shortest n -link billiard trajectories from A to B .

Proposition 1. *One has: $|P_n - Q_n| = O(1/n^2)$.*

Remark 1. This Claim is consistent with Theorem 6 (iii) of [7] which describes the limit distribution of the vertices of the inscribed polygons that best approximate a convex curve relative the deviation of the perimeter length; see also [5].

Proof. To prove Proposition 1, we use the theory of interpolating Hamiltonians, see [6, 8, 9] and [13]. Recall the relevant facts from this theory.

First, some generalities about plane billiards (see, e. g., [15, 16]). The phase space X of the billiard ball map consists of inward unit tangent vectors (x, v) to M with the foot point x on the boundary ∂M ; x is the position of the billiard ball and v is its velocity. The billiard ball map F takes (x, v) to the vector obtained by moving x along v until it hits ∂M and then elastically reflecting v according to the law “angle of incidence equals angle of reflection”. Let ϕ be the angle made by v with the positive direction of ∂M . Then (s, ϕ) are coordinates in X . The area form $\omega = \sin \phi \, d\phi \wedge ds$ is F -invariant.

In a nutshell, the theory of interpolating Hamiltonians asserts that the billiard ball map equals an integrable symplectic map, modulo smooth symplectic maps that fix the boundary of the phase space X to all orders. More specifically, one can choose new symplectic coordinates H and Z near the boundary $\phi = 0$ such that $\omega = dH \wedge dZ$, H is an integral of the map F , up to all orders in ϕ , and

$$F^*(Z) = Z + H^{1/2}, \quad (1)$$

also up to all orders in ϕ . The function H is given by a series in even powers of ϕ , namely,

$$H = k^{-2/3}\phi^2 + O(\phi^4), \quad (2)$$

and this series is uniquely determined by the above conditions on H and Z .

Lemma 2.1. *One may choose the coordinate Z in such a way that $Z = \sigma + O(\phi^2)$.*

Proof. Let $Z = f(s) + g(s)\phi + O(\phi^2)$. We have: $\omega = dH \wedge dZ$. Equating the coefficients of $\phi \, d\phi \wedge ds$ and of $\phi^2 \, d\phi \wedge ds$ and using (2) we obtain the equations:

$$2k^{-2/3}(s)f'(s) = 1, \quad 2k^{-2/3}(s)g'(s) + \frac{2}{3}k^{-5/3}(s)k'(s)g(s) = 0.$$

The first equation implies that $df = d\sigma$ and the second that $g = Ck^{-1/3}$ where C is a constant. We can choose $f(0) = 0$. Since Z is defined up to summation with functions of H , it follows from (2) that the term $g(s)\phi$ can be eliminated by subtracting $CH^{1/2}$. \square

Now we can prove Proposition 1. The billiard trajectory T_n corresponds to a phase orbit x_0, \dots, x_n , $F(x_i) = x_{i+1}$. Since H is an integral of the map F , the orbit x_0, \dots, x_n lies on a level curve $H = c_n$. Due to (1), we have: $n\sqrt{c_n} = O(1)$, and hence $c_n = O(1/n^2)$ which, in view of (2), implies that

$$\phi = O\left(\frac{1}{n}\right). \quad (3)$$

Consider σ and Z as functions on the phase space X . Since $\sigma(x_m) = P_m$ and the σ -coordinate of Q_m is m/n , we need to show that

$$\sigma(x_m) = \frac{m}{n} + O\left(\frac{1}{n^2}\right). \tag{4}$$

Since F is a shift in Z -coordinate, see (1), one has:

$$Z(x_m) = \frac{m}{n} (Z(x_n) - Z(x_0)) = \frac{m}{n} (\sigma(x_n) - \sigma(x_0)) + O\left(\frac{1}{n^2}\right) = \frac{m}{n} + O\left(\frac{1}{n^2}\right),$$

the second equality due to Lemma 2.1 and (3). This proves Proposition 1. \square

From now on, we identify the arc γ with the segment $[0, 1]$ using the parameter σ ; the points P_1, \dots, P_{n-1} are considered as reals between 0 and 1. We have a finite set $S' = \{t_1, \dots, t_k\} \subset (0, 1)$. Some of the numbers $t_i \in S'$ may be rational; denote them by p_i/q_i , $i = 1, \dots, l$ (fractions in lowest terms), and let $Q = q_1 \cdots q_l$. Set $n_i(N) = 1 + (N + i)Q$, $i = 0, \dots, k$; $N \in \mathbb{Z}$.

Proposition 2. *For N sufficiently large, at least one of the sets $\mathbf{P}_{n_i(N)}$ is disjoint from S' .*

Proof. Assume not. Then, by the Pigeonhole Principle, there exist l, i, j such that $t_l \in \mathbf{P}_{n_i} \cap \mathbf{P}_{n_j}$ (we suppress N in the notation $n_i(N)$). According to Proposition 1, there is a constant C (independent of n) such that, for $P_m \in \mathbf{P}_n$, one has:

$$\left|P_m - \frac{m}{n}\right| < \frac{C}{n^2}.$$

Therefore

$$\left|t_l - \frac{m_1}{n_i}\right| < \frac{C}{n_i^2}, \quad \left|t_l - \frac{m_2}{n_j}\right| < \frac{C}{n_j^2} \tag{5}$$

for some m_1, m_2 .

Lemma 2.2. *If N sufficiently large then $t_i \notin \mathbb{Q}$.*

Proof. First, we claim that, given a fraction p/q and a constant C , if

$$\left|\frac{p}{q} - \frac{m}{n}\right| < \frac{C}{n^2}$$

for all sufficiently large n then $m/n = p/q$.

Indeed, if $m/n \neq p/q$ then $1 \leq |pn - qm|$, hence

$$\frac{1}{qn} \leq \left|\frac{p}{q} - \frac{m}{n}\right| < \frac{C}{n^2},$$

which cannot hold for $n > Cq$.

Next, we claim that, for all $M, N \in \mathbb{Z}$ and each $i = 1, \dots, l$,

$$\frac{M}{1 + NQ} \neq \frac{p_i}{q_i}.$$

Indeed, if the equality holds then $Mq_i = p_i(1 + NQ)$. The right hand side is divisible by q_i but $1 + NQ$ is coprime with q_i ; this contradicts the assumption that q_i and p_i are coprime.

The two claims combined imply the lemma. \square

Next, (5) and the triangle inequality imply that

$$\left| \frac{m_1}{n_i} - \frac{m_2}{n_j} \right| < C \left(\frac{1}{n_i^2} + \frac{1}{n_j^2} \right)$$

for some m_1, m_2 . It follows that

$$|m_1 n_j - m_2 n_i| < C \left(\frac{n_j}{n_i} + \frac{n_i}{n_j} \right).$$

The expression in the parentheses on the right hand side has limit 2, as $N \rightarrow \infty$, hence one has, for sufficiently great N ,

$$|m_1 n_j - m_2 n_i| < 3C. \tag{6}$$

Denote by \mathbf{M} the (finite) set of fractions with the denominators jQ , $j \in \{1, 2, \dots, k\}$, and let $\delta > 0$ be the distance between the sets $S' - \mathbb{Q}$ and \mathbf{M} .

Lemma 2.3. *For sufficiently large N , one has:*

$$|m_1 n_j - m_2 n_i| > \delta Q^2 N / 2.$$

Proof. For N large enough, it follows from (5) that

$$\left| t_l - \frac{m_1}{n_i} \right| < \frac{\delta}{2}.$$

Since $t_l \notin \mathbb{Q}$, it follows that the distance from m_1/n_i to \mathbf{M} is greater than $\delta/2$. One has:

$$|m_1 n_j - m_2 n_i| = |n_j - n_i| n_i \left| \frac{m_1}{n_i} - \frac{m_2 - m_1}{n_j - n_i} \right| > Q \cdot QN \cdot \frac{\delta}{2},$$

as claimed. □

Finally, for large N , Lemma 2.3 contradicts inequality (6), and Proposition 2 follows. □

Let us put all things together. Recall that \mathbf{P}_m denotes the set of vertices of the shortest m -link billiard trajectory from point A to point B . We introduced, before the formulation of Proposition 2, $k + 1$ arithmetic progressions $n_i(N)$, $i = 0, \dots, k$; $N \in \mathbb{Z}$. Proposition 2 asserts that for any finite subset S' of the arc AB there exists N_0 such that for each $N > N_0$ at least one of the $k + 1$ sets $\mathbf{P}_{n_i(N)}$ is disjoint from S' . Thus S' does not block B from A . Furthermore, for any finite set S'' inside the billiard table, there exists N_1 such that for each $m > N_1$ the shortest m -link billiard trajectory from A to B is disjoint from S'' . Therefore the set $S = S' \cup S''$ does not block B from A , and Theorem 1.1 follows.

Remark 2. Theorem 1.1 can be extended to billiards in higher dimensional Euclidean spaces. The argument is essentially the same; we sketch it below.

Let A and B be sufficiently close points on a smooth strictly convex part of the boundary of the billiard table and let γ be the shortest geodesic connecting A and B . Let $S = S' \cup S''$ be a finite set of points where S' lie on γ and S'' off γ . First, for n large enough, the shortest n -link billiard trajectory from A to B lies in arbitrarily small neighborhood of γ . Thus the set S'' does not block B from A . Secondly, an analog of Proposition 1 holds: in the limit $n \rightarrow \infty$, the vertices of this n -link billiard trajectory get uniformly distributed with respect to the measure $k^{2/3} ds$, where s is an arc-length parameter on γ and k is its normal curvature; the error is of order

$1/n^2$. Therefore the set S' does not block B from A either. Both claims follow from the theory of interpolating Hamiltonians as described in detail in sections 7.2–7.3 of [13].

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3. Appendix: a result on rational approximation, by R. Schwartz. The purpose of this appendix is to establish Theorem 3.2 below. This theorem improves on the estimates in Section 2 and, for $f(n) = 1/n$, implies them. We don't know if this is a new result. As usual, \mathbb{N} denotes the set of natural numbers.

Until the statement of Theorem 3.2, we fix an irrational $x \in (0, 1)$. Let $f : \mathbb{N} \rightarrow (0, \infty)$ be a bounded function. Say that $n \in \mathbb{N}$ is *good* relative to f if there is some $m \in \mathbb{N}$ such that

$$\left| x - \frac{m}{n} \right| < \frac{f(n)}{n}; \quad (7)$$

m and n need not be relatively prime. Let $G(f)$ denote the set of good integers. Let $\Delta G(f)$ denote the upper density of $G(f)$.

Let f_ε be the function such that $f_\varepsilon(n) = \varepsilon$ for all $n \in \mathbb{N}$.

Lemma 3.1. *If $\varepsilon < 1/2$ then $\Delta G(f_\varepsilon) = 2\varepsilon$.*

Proof. Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle. Let $I \subset S^1$ denote the interval of length 2ε centered about 0. Let $\phi : S^1 \rightarrow S^1$ be rotation by x . Then $n \in G(f)$ if and only if $\phi^n(0) \in I$. By the Ergodic Theorem the set of n having this property has density, and hence upper density, equal to $|I|$. \square

Corollary 1. *If $\lim_{r \rightarrow \infty} f(r) = 0$ then $\Delta G(f) = 0$.*

Proof. For any $\varepsilon > 0$ we have $f < f_\varepsilon$ on the complement of a finite subset of \mathbb{N} . Hence $G(f) - G(f_\varepsilon)$ is a finite set. Hence $\Delta G(f) \leq \Delta G(f_\varepsilon) = 2\varepsilon$. Since ε is arbitrary, we have $\Delta G(f) = 0$. \square

Theorem 3.2. *Let f be any function such that $\lim_{r \rightarrow \infty} f(r) = 0$. Let x_1, \dots, x_v be any finite set of irrational numbers in $(0, 1)$. Then, for every N , there exists $n > N$ such that*

$$\left| \frac{i}{n} - x_j \right| \geq \frac{f(n)}{n}; \quad \forall (i, j) \in \{1, \dots, n-1\} \times \{1, \dots, v\}.$$

Proof. Let $G_j(f)$ be as above, relative to x_j . If this result is false then $\bigcup_{j=1}^v G_j(f) = \mathbb{N}$. But then $0 = \sum_{j=1}^v \Delta G_j(f) \geq 1$, contradiction. \square

REFERENCES

- [1] K. Burns and E. Gutkin, *Growth of the number of geodesics between points and insecurity for riemannian manifolds*, Discrete Contin. Dyn. Syst., **21** (2008), 403–413.
- [2] E. Gutkin, *Blocking of billiard orbits and security for polygons and flat surfaces*, Geom. Funct. Anal., **15** (2005), 83–105.
- [3] E. Gutkin and V. Schroeder, *Connecting geodesics and security of configurations in compact locally symmetric spaces*, Geom. Dedicata, **118** (2006), 185–208.
- [4] J.-F. Lafont and B. Schmidt, *Blocking light in compact Riemannian manifolds*, Geometry and Topology, **11** (2007), 867–887.
- [5] M. Ludwig, *Asymptotic approximation of convex curves*, Arch. Math., **63** (1994), 377–384.
- [6] S. Marvizi and R. Melrose, *Spectral invariants of convex planar regions*, J. Diff. Geom., **17** (1982), 475–502.

- [7] D. McClure and R. Vitale, *Polygonal approximation of plane convex bodies*, J. Math. Anal. Appl., **51** (1975), 326–358.
- [8] R. Melrose, *Equivalence of glancing hypersurfaces*, Invent. Math., **37** (1976), 165–192.
- [9] R. Melrose, *Equivalence of glancing hypersurfaces 2*, Math. Ann., **255** (1981), 159–198.
- [10] T. Monteil, *A counter-example to the theorem of Hiemer and Snurnikov*, J. Statist. Phys., **114** (2004), 1619–1623.
- [11] T. Monteil, *On the finite blocking property*, Ann. Inst. Fourier, **55** (2005), 1195–1217.
- [12] T. Monteil, *Finite blocking property versus pure periodicity*, preprint, [arXiv:math/0406506](https://arxiv.org/abs/math/0406506)
- [13] V. Petkov and L. Stoyanov, “Geometry of Reflecting Rays and Inverse Spectral Problems,” John Wiley & Sons, Chichester, 1992.
- [14] B. Schmidt and J. Souto, *Chords, light, and another synthetic characterization of the round sphere*, preprint.
- [15] S. Tabachnikov, “Billiards,” Soc. Math. de France, Paris, 1995.
- [16] S. Tabachnikov, “Geometry and Billiards,” Amer. Math. Soc., Providence, RI, 2005.

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