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Topological Robotics: Motion Planning in Projective Spaces

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1 Introduction

In this paper, we study one of the most elementary problems of the topological robotics: rotation of a line, which is fixed by a revolving joint at a base point. One wants to bring the line from its initial position A to a final position B by a continuous motion in space. The ultimate goal is to construct a motion planning algorithm which will perform this task once the initial position A and the final position B are presented. This problem becomes hard when the dimension of the space is large.

Any such motion planning algorithm must have instabilities, that is, the motion of the system will be discontinuous as a function of A and B. These instabilities are caused by topological reasons. A general approach to study instabilities of robot motion was suggested recently in [6, 7]. With any path-connected topological space X, one associates in [6, 7] a number $\mathbf{TC}(X)$, called *the topological complexity of* X. This number is of fundamental importance for the motion planning problem: $\mathbf{TC}(X)$ determines character of instabilities which have all motion planning algorithms in X.

The motion planning problem of moving a line in $\mathbb{R}^{n+1}$ reduces to a topological problem of calculating the topological complexity of the real projective space $\mathbf{TC}(\mathbb{RP}^n)$, which we tackle in this paper. We compute the number $\mathbf{TC}(\mathbb{RP}^n)$ for all $n \leq 23$ (see Table 8.1). This will probably be useful for applications.

Surprisingly, the problem of finding a general formula for $\mathbf{TC}(\mathbb{RP}^n)$ turns out to be quite difficult. One of the main results of this paper claims that, for $n \neq 1, 3, 7$, the

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number $\mathbf{TC}(\mathbb{RP}^n)$ coincides with the smallest integer k such that the projective space $\mathbb{RP}^n$ admits an immersion into $\mathbb{R}^{k-1}$. This means that our problem of finding $\mathbf{TC}(\mathbb{RP}^n)$ is equivalent to the immersion problem for real projective spaces. The latter is a classical well-researched topological problem, see [4, 9] for a survey. A general answer to the immersion problem $\mathbb{RP}^n \to \mathbb{R}^{k-1}$ is not known, but there are many important results in the literature.

For the remarkable special values $n = 1, 3, 7$, the answer is very simple: $\mathbf{TC}(\mathbb{RP}^n) = n + 1$.

We show that for $n \leq 8$ explicit motion planning algorithms for moving lines in $\mathbb{R}^n$ can be built by using multiplication of the complex numbers, the quaternions, and the Cayley numbers.

2 The motion planning problem

In this section, we recall some definitions and results from [6, 7].

Let X be a topological space, thought of as the configuration space of a mechanical system. We always assume that X is a connected CW complex. Given two points $A, B \in X$, one wants to connect them by a path in X; this path represents a continuous motion of the system from one configuration to the other. A solution to this motionplanning problem is a rule (algorithm) that takes $(A, B) \in X \times X$ as an input and produces a path from A to B as an output. Let PX denote the space of all continuous paths $\gamma : [0, 1] \to X$, equipped with the compact-open topology, and let $\pi : PX \to X \times X$ be the map assigning the end points to a path: $\pi(\gamma) = (\gamma(0), \gamma(1))$. The map π is a fibration whose fiber is the based loop space ΩX. The motion planning problem consists of finding a section s of this fibration.

The section s cannot be continuous unless X is contractible, see [7]. One defines $\mathbf{TC}(X)$, the topological complexity of X, as the smallest number k such that $X \times X$ can be covered by open sets $U_1, \dots, U_k$ so that for every $i = 1, \dots, k$, there exists a continuous section $s_i : U_i \to PX$, $\pi \circ s_i = \mathrm{id}$.

According to [6], *a motion planner* in X is defined by finitely many subsets $F_1, \dots, F_k \subset X \times X$ and continuous maps $s_i : F_i \to PX$, where $i = 1, \dots, k$, such that

(a) the sets $F_1, \dots, F_k$ are pairwise disjoint $F_i \cap F_j = \varnothing$, $i \neq j$, and cover $X \times X$;
(b) $\pi \circ s_i = 1_{F_i}$ for any $i = 1, \dots, k$;
(c) each F_i is Euclidean neighbourhood retract (ENR).

The subsets F_i are *local domains* of the motion planner; the maps s_i are *local rules*. Any motion planner determines a *motion planning algorithm*: given a pair (A, B) of initial-final configurations, we determine the index $i \in \{1, 2, \dots, k\}$ such that the local domain F_i contains (A, B); then we apply the local rule s_i and produce the path $s_i(A, B) \in PX$ in X as an output.

The order of instability of a motion planner is defined as the largest r such that the closures of some r among the local domains $F_1, \ldots, F_k$ have a nonempty intersection

$$\bar{F}_{i_1} \cap \bar{F}_{i_2} \cap \cdots \cap \bar{F}_{i_r} \neq \varnothing, \quad 1 \leq i_1 < i_2 < \cdots < i_r \leq k. \tag{2.1}$$

The order of instability describes the character of discontinuity of the motion planning algorithm determined by the motion planner.

In [6], it is shown that the minimal integer k, such that a smooth manifold X admits a motion planner with k local rules, equals $\mathbf{TC}(X)$. Moreover, the minimal integer $r > 0$, such that X admits a motion planner with order of instability r, equals $\mathbf{TC}(X)$.

This explains the importance of knowing the number $\mathbf{TC}(X)$ while solving practical motion planning problems.

We mention some other results from [7], which will be used later in this paper. The number $\mathbf{TC}(X)$ depends only on the homotopy type of X. One has

$$\operatorname{cat}(X) \leq \mathbf{TC}(X) \leq 2\operatorname{cat}(X) - 1, \tag{2.2}$$

where $\operatorname{cat}(X)$ is the Lusternik-Schnirelmann category of X. If X is r-connected, then

$$\mathbf{TC}(X) < \frac{2\dim(X) + 1}{r + 1} + 1. \tag{2.3}$$

Next result provides a lower bound for $\mathbf{TC}(X)$ in terms of the cohomology ring $H^*(X)$ with coefficients in a field. The tensor product $H^*(X) \otimes H^*(X)$ is also a graded ring with the multiplication

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (-1)^{|v_1| \cdot |u_2|} u_1 u_2 \otimes v_1 v_2, \tag{2.4}$$

where $|v_1|$ and $|u_2|$ are the degrees of the cohomology classes v_1 and u_2. The cohomology multiplication $H^*(X) \otimes H^*(X) \to H^*(X)$ is a ring homomorphism. Let $I \subset H^*(X) \otimes H^*(X)$ be the kernel of this homomorphism. The ideal I is called *the ideal of zero-divisors* of $H^*(X)$. The *zero-divisors-cup-length* is the length of the longest nontrivial product in the ideal of zero-divisors. It is shown in [7] that *the topological complexity* $\mathbf{TC}(X)$ *is greater than the zero-divisors-cup-length of* $H^*(X)$.

In this estimate, one can equally well use extraordinary cohomology theories instead of the usual cohomology.

The topological complexity $\mathbf{TC}(X)$ as well as the Lusternik-Schnirelmann category $\mathrm{cat}(X)$ are particular cases of the notion of *Schwarz genus* (also known as *sectional category*) of a fibration; it was introduced and thoroughly studied by Schwarz in [15], see also [10, 11].

3 Motion planning in simply connected symplectic manifolds

Theorem 3.1. Let X be a simply connected $2n$-dimensional finite polyhedron. Suppose that there exists a cohomology class $u \in H^2(X;\mathbf{k})$, where $\mathbf{k}$ is a field, such that $u^n \neq 0 \in H^{2n}(X;\mathbf{k})$. Then $\mathbf{TC}(X) = 2n+1$. ☐

Proof. Since X is simply connected, the inequality $\mathbf{TC}(X) \leq 2n+1$ follows from (2.3). On the other hand, $u \otimes 1 - 1 \otimes u \in H^*(X;\mathbf{k}) \otimes H^*(X;\mathbf{k})$ is a zero-divisor (see the definition in Section 2) whose $2n$th power

$$(u \otimes 1 - 1 \otimes u)^{2n} = (-1)^n \binom{2n}{n} u^n \otimes u^n \tag{3.1}$$

does not vanish. The opposite inequality $\mathbf{TC}(X) \geq 2n+1$ now follows from [7, Theorem 7]. ■

Corollary 3.2. If X is a closed simply connected symplectic manifold, then

$$\mathbf{TC}(X) = \dim(X) + 1. \tag{3.2}$$

☐

In particular,

$$\mathbf{TC}\left(\mathbb{CP}^n\right) = 2n+1. \tag{3.3}$$

4 Motion planning in the real projective space: the initial discussion

In this section, we start studying the problem of computing the topological complexity of the real projective space $\mathbf{TC}(\mathbb{RP}^n)$. This problem is much harder than finding the topological complexity of the complex projective space (given by (3.3)). We show that the problem of finding the number $\mathbf{TC}(\mathbb{RP}^n)$ is equivalent to the problem of finding the smallest k such that $\mathbb{RP}^n$ can be immersed into the Euclidean space $\mathbb{R}^k$.

We begin this section by proving a general result relating the topological complexity of a topological space with the Schwarz genus of a covering.

Theorem 4.1. Let X be a finite-connected polyhedron and let $p : \tilde{X} \to X$ be a regular covering map with the group of covering transformations G. Let $\tilde{X} \times_G \tilde{X}$ be obtained from the product $\tilde{X} \times \tilde{X}$ by factorizing with respect to the diagonal action of G. Then, the topological complexity $\mathbf{TC}(X)$ of the space X is greater than or equal to the Schwarz genus of the covering

$$q : \tilde{X} \times_G \tilde{X} \longrightarrow X \times X. \tag{4.1}$$

□

Proof. We use the notation from Section 2. In particular, $\pi : PX \to X \times X$ will denote the canonical fibration of the space of paths $\pi(\gamma) = (\gamma(0), \gamma(1))$, where $\gamma \in PX$, $\gamma : [0,1] \to X$.

Consider the following commutative diagram:

$$\begin{tikzcd} PX \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \tilde{X} \times_G \tilde{X} \arrow[dl, "q"] \\ & X \times X, & \end{tikzcd} \tag{4.2}$$

where the map $f : PX \to \tilde{X} \times_G \tilde{X}$ is defined as follows: given a continuous path $\gamma : [0,1] \to X$, let $\tilde{\gamma} : [0,1] \to \tilde{X}$ be any lift of γ, and we set

$$f(\gamma) = \big(\tilde{\gamma}(0), \tilde{\gamma}(1)\big) \in \tilde{X} \times_G \tilde{X}. \tag{4.3}$$

The lift $\tilde{\gamma}$ of γ depends on the choice of the initial point $\tilde{\gamma}(0) \in \tilde{X}$ but nevertheless the map f is well defined and continuous. If U is an open subset of $X \times X$ and $s : U \to PX$ is a continuous section of the fibration π over U, then $f \circ s$ is a continuous section of q over U. If there exits an open covering $U_1 \cup U_2 \cup \cdots \cup U_k$ of $X \times X$ with a continuous section s_i of π over each open set U_i, then $f \circ s_i$ is a continuous section of q over U_i and we see that the Schwarz genus of the covering q is at most k. ■

Remark 4.2. In general, (4.1) is not a regular covering; it is regular if and only if the group G is Abelian.

Next, we are going to apply Theorem 4.1 to the case $X = \mathbb{RP}^n$.

Corollary 4.3. The number $\mathbf{TC}(\mathbb{RP}^n)$ is greater than or equal to the Schwarz genus of the two-fold covering $S^n \times_{\mathbb{Z}_2} S^n \to \mathbb{RP}^n \times \mathbb{RP}^n$. □

We present this result in a different form.

If n is fixed, we always denote by ξ the canonical real line bundle over $\mathbb{RP}^n$. The exterior tensor product $\xi \otimes \xi$ is a real line bundle over $\mathbb{RP}^n \times \mathbb{RP}^n$. Its first Stiefel-Whitney class is

$$w_1(\xi \otimes \xi) = \alpha \times 1 + 1 \times \alpha \in H^1\left(\mathbb{RP}^n \times \mathbb{RP}^n; \mathbb{Z}_2\right), \tag{4.4}$$

where $\alpha \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ is the generator. This last condition determines uniquely the bundle $\xi \otimes \xi$.

Corollary 4.4. The topological complexity $\mathbf{TC}(\mathbb{RP}^n)$ is not less than the minimal k such that the Whitney sum $k(\xi \otimes \xi)$ of k copies of $\xi \otimes \xi$ admits a nowhere vanishing section. □

Proof. By Theorem 4.1, $\mathbf{TC}(\mathbb{RP}^n)$ is not less than the Schwarz genus of the unit sphere bundle q of $\xi \otimes \xi$. By Theorem of Schwarz [15], the latter coincides with the smallest k such that the k-fold fiberwise join $q * q * \cdots * q$ admits a section. But, clearly, the k-fold fiberwise join $q * q * \cdots * q$ coincides with the unit sphere bundle of the Whitney sum $k(\xi \otimes \xi)$. This implies our statement. ■

We show below that the lower bound given by Corollaries 4.3 and 4.4 are in fact sharp.

The left-hand side inequality in (2.2) gives

$$\mathbf{TC}\left(\mathbb{RP}^n\right) \geq n+1, \tag{4.5}$$

since the Lusternik-Schnirelman category of $\mathbb{RP}^n$ is $n+1$. We show later in this paper that an equality holds in (4.5) if and only if $n = 1, 3, 7$.

Theorem 4.5. If $n \geq 2^{r-1}$, then $\mathbf{TC}(\mathbb{RP}^n) \geq 2^r$. □

Proof. Let $\alpha \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ be the generator. Then, $1 \otimes \alpha + \alpha \otimes 1$ is a zero-divisor (see Section 2), and we consider its power

$$(1 \otimes \alpha + \alpha \otimes 1)^{2^r - 1}. \tag{4.6}$$

The binomial expansion of this class contains the term

$$\binom{2^r - 1}{n} \alpha^k \otimes \alpha^n, \tag{4.7}$$

where $k = 2^r - 1 - n < n$. It is well known that the binomial coefficients $\binom{2^r-1}{i}$ are odd for all i. Hence (4.7) is a nonzero term, and so (4.6) does not vanish either. Applying [7, Theorem 7], one finds that the topological complexity of $\mathbb{RP}^n$ is not less than 2^r. ■

Remark 4.6. A different proof of Theorem 4.5 can be based on Corollary 4.4. Let k_n denote the least k such that the top Stiefel-Whitney class w_k of $k(\xi \otimes \xi)$ vanishes. Then, $\mathbf{TC}(\mathbb{RP}^n) \geq k_n$ (since the top Stiefel-Whitney class of a vector bundle having a section, vanishes). We want to show that $k_n \geq 2^r$ for $n \geq 2^{r-1}$. By Cartan's formula, $w_k(k(\xi \otimes \xi)) = (\alpha_1 + \alpha_2)^k$, where $\alpha_1, \alpha_2 \in H^1(\mathbb{RP}^n \times \mathbb{RP}^n; \mathbb{Z}_2)$ denote the standard 1-dimensional generators. If $k = 2^r - 1$, then all binomial coefficients $\binom{k}{i}$ are odd and since $2^r - 1 \leq 2n$, the Stiefel-Whitney class $w_k(k(\xi \otimes \xi))$ contains a nontrivial term $\binom{k}{i}\alpha_1^i \alpha_2^{k-i}$. Hence $k_n \geq 2^r$ and the result follows.

5 Nonsingular maps and axial maps

In this section, we recall some notions and basic results concerning nonsingular maps $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ and axial maps $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$. These maps appear in the mathematical literature in relation to the immersion problem $\mathbb{RP}^n \to \mathbb{R}^k$, see below. The material of this section (except maybe Lemma 5.7) should be considered as well known.

Definition 5.1. A continuous map

$$f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^k \tag{5.1}$$

is called nonsingular if it has the following two properties:

(a) $f(\lambda u, \mu v) = \lambda\mu f(u, v)$ for all $u, v \in \mathbb{R}^n$, $\lambda, \mu \in \mathbb{R}$,

(b) $f(u, v) = 0$ implies that either $u = 0$ or $v = 0$.

Our definition of a nonsingular map is not quite standard, we do not require f to be bilinear. Bilinear nonsingular maps give immersions of projective spaces into Euclidean space, see [8]. Constructions of bilinear nonsingular maps are given in [12, 14]. In [13], Lam considers maps $f(u, v)$ with property (b), which are linear only with respect to v and satisfy a weaker property $f(-u, v) = -f(u, v)$ with respect to u; he calls such maps skew-linear.

We show (see proof of Proposition 6.3) that the nonsingular maps in the sense of Definition 5.1 provide a convenient tool for constructing explicit motion planning algorithms in projective spaces.

As an illustration, let us show that for any n there exists a nonsingular map $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{2n-1}$. One constructs it as follows. Fix a sequence $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1} : \mathbb{R}^n \to \mathbb{R}$ of linear functionals such that any n of them are linearly independent. For $u, v \in \mathbb{R}^n$, the value $f(u, v) \in \mathbb{R}^{2n-1}$ is defined as the vector whose jth coordinate equals the product $\alpha_j(u)\alpha_j(v)$, where $j = 1, 2, \ldots, 2n-1$. If $u \neq 0$, then at least n among the numbers $\alpha_1(u), \ldots, \alpha_{2n-1}(u)$ are nonzero. Hence if $u \neq 0$ and $v \neq 0$, there exists j such that $\alpha_j(u)\alpha_j(v) \neq 0$ and thus $f(u, v) \neq 0 \in \mathbb{R}^{2n-1}$.

Lemma 5.2. There are no nonsingular maps $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ with $k < n$. □

Proof. We may apply the Borsuk-Ulam theorem to the map $u \mapsto f(u, v)$, where $v \neq 0$ is fixed and where u varies on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. By the Borsuk-Ulam theorem, $f(u, v) = f(-u, v)$ for some $u \in S^{n-1}$, but the latter also is $-f(u, v)$ and thus $f(u, v) = 0$. This gives a contradiction with the nonsingularity property. ■

Lemma 5.3. For $n = 1, 2, 4$, or 8, there exists a nonsingular map $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with the property that for any $u \in \mathbb{R}^n$, $u \neq 0$, the first coordinate of $f(u, u)$ is positive. □

Proof. For $n = 1$, we take $f(u, v) = uv$, the usual product of real numbers. For $n = 2$, we may take $f(u, v) = u\bar{v}$, the product of u and the conjugate of v, viewed as complex numbers.

We identify $\mathbb{R}^4$ with the set of quaternions $v = x_1 + x_2 i + x_3 j + x_4 k$ and for $n = 4$, define the nonsingular map f by $f(u, v) = u\bar{v} \in \mathbb{R}^4$, where the bar denotes the quaternionic conjugation $\bar{v} = x_1 - x_2 i - x_3 j - x_4 k$.

To construct f for $n = 8$, we identify $\mathbb{R}^8$ with the ring of Cayley numbers, which can be defined as follows, see [5]. A Cayley number can be uniquely written in the form $q + Qe$, where q and Q are quaternions and e is a formal symbol. The multiplication is defined by the formula

$$(q + Qe) \cdot (r + Re) = (qr - \bar{R}Q) + (Rq + Q\bar{r})e. \tag{5.2}$$

Here, the bar denotes the conjugation of quaternions as above. We define $f : \mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}^8$ by

$$f(q + Qe, r + Re) = (q + Qe) \cdot (\bar{r} - Re). \tag{5.3}$$

Then, $f(q + Qe, q + Qe) = q\bar{q} + Q\bar{Q}$ is real and positive, assuming that the Cayley number $q + Qe \neq 0$ is nonzero. ■

Lemma 5.4. For n distinct from $1, 2, 4, 8$, there are no nonsingular maps $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. □

Proof. The statement can be deduced from the Adams' solution of the Hopf invariant one problem [1]. Namely, suppose that f as above exists, where $n > 2$. Then, we obtain a continuous map $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ given by

$$g(x,y) = \frac{f(x,y)}{|f(x,y)|}, \quad x,y \in S^{n-1}. \tag{5.4}$$

The map g satisfies $g(-x,y) = -g(x,y) = g(x,-y)$. Restricting g onto one factor $S^{n-1} \times *$ (where $*$ is a base point) gives a self map of S^{n-1} which commutes with the antipodal involution and hence has an odd degree (this is a theorem of Borsuk, see [3, pages 483–485]). Similarly, the degree of $g|_{*\times S^{n-1}}$ is odd. Hence, the bidegree of g is (k,ℓ), where both integers k and ℓ are odd. The existence of g implies the vanishing of the Whitehead product

$$[k\iota, \ell\iota] = k\ell[\iota,\iota] \in \pi_{2n-3}S^{n-1}, \tag{5.5}$$

where $\iota \in \pi_{n-1}S^{n-1}$ is the generator. If n is odd, then $[\iota,\iota] \in \pi_{2n-3}S^{n-1}$ is of infinite order, hence the Whitehead product (5.5) cannot vanish. If n is even and distinct from $1, 2, 4, 8$, then the Whitehead product $[\iota,\iota] \in \pi_{2n-3}S^{n-1}$ is nonzero (as proven by Adams) and has order two, which again implies that (5.5) is nonzero as $k\ell$ is odd. ■

Definition 5.5. Let n and k be two positive integers with $n < k$. A continuous map

$$g : \mathbb{RP}^n \times \mathbb{RP}^n \longrightarrow \mathbb{RP}^k \tag{5.6}$$

is called axial of type (n,k) if its restrictions to $* \times \mathbb{RP}^n$ and $\mathbb{RP}^n \times *$ are homotopic to the inclusions maps $\mathbb{RP}^n \to \mathbb{RP}^k$.

Here, $*$ denotes a base point of $\mathbb{RP}^n$. Note that for $n < k$ any continuous map $h : \mathbb{RP}^n \to \mathbb{RP}^k$ is either homotopically trivial or it is homotopic to the inclusion map $\mathbb{RP}^n \to \mathbb{RP}^k$. If $\alpha \in H^1(\mathbb{RP}^k;\mathbb{Z}_2)$ denotes the generator, then $h^*\alpha \in H^1(\mathbb{RP}^n;\mathbb{Z}_2)$ is either zero or equal to α. The map h is homotopically trivial if and only if $h^*\alpha = 0$. This shows that the property of the axial map g, see (5.6), can be equivalently stated by the formula $g^*\alpha = \alpha \times 1 + 1 \times \alpha$. This last condition fixes the homotopy type of a map $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^\infty$ and we are interested in finding the smallest k such that this map can be factorized through the inclusion $\mathbb{RP}^k \to \mathbb{RP}^\infty$. This discussion explains that there always exists an axial map $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^{2n}$. In fact, with some extra effort, one shows that there always exists an axial map $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^{2n-1}$.

Lemma 5.6. Assume that $1 < n < k$. There exists a bijection between nonsingular maps $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ (viewed up to multiplication by a nonzero scalar) and axial maps $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$. □

Proof. Given a nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$, one defines $g : \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$, where for $u, v \in S^n \subset \mathbb{R}^{n+1}$, the value $g(u, v)$ is the line through the origin containing the point $f(u, v) \in \mathbb{R}^{k+1}$. To show that g is indeed axial, we fix $v \in S^n$ and vary only $u \in S^n$. We see that the obtained map $\mathbb{RP}^n \to \mathbb{RP}^k$ lifts to a map $S^n \to S^k$ given by $u \mapsto f(u, v)$, and the relation $f(-u, v) = -f(u, v)$ implies that $\mathbb{RP}^n \to \mathbb{RP}^k$ is not null-homotopic. Similarly, using $f(u, -v) = -f(u, v)$, we find that the restriction of g onto $* \times \mathbb{RP}^n$ is not null-homotopic.

Suppose now that we are given an axial map (5.6). Passing to the universal covers, we obtain a continuous map $\bar{g} : S^n \times S^n \to S^k$ (defined up to a sign). As explained above, the axial property translates into $\bar{g}(-u, v) = -\bar{g}(u, v) = \bar{g}(u, -v)$ for all $u, v \in S^n$. Now, we may define a nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ by

$$f(u, v) = |u| \cdot |v| \cdot \bar{g}\left(\frac{u}{|u|}, \frac{v}{|v|}\right), \quad u, v \in \mathbb{R}^{k+1} - \{0\}, \tag{5.7}$$

completing the proof. ■

Lemma 5.7. Suppose that for a pair of integers $1 < n < k$, there exists a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$. Then, there exists a nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$, having the following additional property: for any $u \in \mathbb{R}^{n+1}$, $u \neq 0$, the first coordinate of $f(u, u) \in \mathbb{R}^{k+1}$ is positive. □

Proof. Given a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$, consider the corresponding axial map $g : \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$, see Lemma 5.6. The axial property implies that the restriction of g onto the diagonal $\mathbb{RP}^n \subset \mathbb{RP}^n \times \mathbb{RP}^n$ is null-homotopic. Hence, we may find $g' \simeq g$ such that $g' : \mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$ is constant along the diagonal. Now, consider the nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ corresponding to g' by Lemma 5.6. We see that for all $u \in \mathbb{R}^{k+1}$, the values $f(u, u) \in \mathbb{R}^{k+1}$ lie on a ray emanating from the origin. By performing an orthogonal rotation, we may assume that all nonzero vectors of this ray have positive first coordinates. This proves our claim. ■

6 The main theorem

Theorem 6.1. The number $\mathbf{TC}(\mathbb{RP}^n)$ coincides with the smallest integer k such that there exists a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$. □

The proof of Theorem 6.1 (see below) uses the following three statements.

Proposition 6.2. For $n > 1$, let k be an integer such that the rank k vector bundle $k(\xi \otimes \xi)$ over $\mathbb{RP}^n \times \mathbb{RP}^n$ (see Section 4) admits a nowhere vanishing section. Then, there exists a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$. $\square$

Proof. Note that the canonical bundle ξ over $\mathbb{RP}^n$ can be represented as the projection $S^n \times_{\mathbb{Z}_2} \mathbb{R} \to S^n/\mathbb{Z}_2$. Here, $\mathbb{Z}_2$ acts as the antipodal involution on S^n and by $x \mapsto -x$ on $\mathbb{R}$. Hence, the bundle $k(\xi \otimes \xi)$ can be represented as the projection

$$\left(S^n \times S^n\right) \times_G \mathbb{R}^k \longrightarrow \left(S^n \times S^n\right)/G, \tag{6.1}$$

where G denotes $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ acting as two antipodal involutions on $S^n \times S^n$ with each summand $\mathbb{Z}_2$ acting as $x \mapsto -x$ on $\mathbb{R}^k$. Consider the following commutative diagram:

$$\begin{tikzcd}
S^n \times S^n \times \mathbb{R}^k \arrow[r, "q_1"'] \arrow[d, "p_1"] & \left(S^n \times S^n \times \mathbb{R}^k\right)/G = E\left(k(\xi \otimes \xi)\right) \arrow[d, "p_2"] \\
S^n \times S^n \arrow[r, "q_2"] & \left(S^n \times S^n\right)/G = \mathbb{RP}^n \times \mathbb{RP}^n.
\end{tikzcd} \tag{6.2}$$

Then q_1 and q_2 are regular G-covers, p_1 is the trivial k-plane bundle, and p_2 can be identified with the bundle $k(\xi \otimes \xi)$.

Any continuous section $s : \mathbb{RP}^n \times \mathbb{RP}^n \to E(k(\xi \otimes \xi))$ of p_2 determines a continuous map $s_1 : S^n \times S^n \to E(k(\xi \times \xi))$ which is invariant under G, that is, $s_1(g(x,y)) = s_1(x,y)$ for any $g \in G$. Since $n > 1$, the space $S^n \times S^n$ is simply connected and so s_1 can be lifted into the covering q_1, which gives a continuous map $s_2 : S^n \times S^n \to S^n \times S^n \times \mathbb{R}^k$. The lift s_2 is not unique (there are exactly four lifts) but there is a unique lift $\tilde{s}_2$ having an additional property $p_1 \circ \tilde{s}_2 = \mathrm{id}$, that is, $\tilde{s}_2$ is a section of the trivial bundle p_1. Then, it is easy to see that for any $g \in G$, one has $g \circ \tilde{s}_2 = \tilde{s}_2 \circ g$, that is, $\tilde{s}_2$ is equivariant.

Let $f : S^n \times S^n \to \mathbb{R}^k - \{0\}$ be the projection of s_2 onto $\mathbb{R}^k$. In other words, $s_2(x,y) = (x, y, f(x,y))$ for $x, y \in S^n$. The result is never zero since the section s assumes no zero values. We have $f(-x,y) = -f(x,y) = f(x,-y)$ for all $x, y \in S^n$. Hence, f determines a nonsingular map $g : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$ given by

$$g(x,y) = f\left(\frac{x}{|x|}, \frac{y}{|y|}\right), \quad x, y \in \mathbb{R}^{n+1}. \tag{6.3}$$

This completes the proof. ■

Proposition 6.3. If there exists a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$, where $n + 1 < k$, then $\mathbb{RP}^n$ admits a motion planner with k local rules, that is,

$$\mathsf{TC}\left(\mathbb{RP}^n\right) \leq k. \tag{6.4}$$

□

Proof. We start with the following observation. Let $\phi : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a scalar continuous map such that $\phi(\lambda u, \mu v) = \lambda\mu\phi(u, v)$ for all $u, v \in V$ and $\lambda, \mu \in \mathbb{R}$. Let $U_\phi \subset \mathbb{RP}^n \times \mathbb{RP}^n$ denote the set of all pairs (A, B) of lines in $\mathbb{R}^{n+1}$ such that $A \neq B$ and $\phi(u, v) \neq 0$ for some points $u \in A$ and $v \in B$. It is clear that U_ϕ is open. We claim that there exists a continuous motion planning strategy over U_ϕ, that is, there is a continuous map s defined on U_ϕ with values in the space of continuous paths in the projective space $\mathbb{RP}^n$ such that, for any pair $(A, B) \in U_\phi$, the path $s(A, B)(t)$, $t \in [0, 1]$ starts at A and ends at B. We may find unit vectors $u \in A$ and $v \in B$ such that $\phi(u, v) > 0$. Such pair u, v is not unique; instead of u, v, we may take $-u, -v$. Note that both pairs u, v and $-u, -v$ determine the same orientation of the plane spanned by A, B. The desired motion planning map s consists in rotating A toward B in this plane, in the positive direction determined by the orientation.

Assume now additionally that $\phi : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is *positive* in the following sense: for any $u \in \mathbb{R}^{n+1}$, $u \neq 0$, one has $\phi(u, u) > 0$. Then, instead of U_ϕ, we may take a slightly larger set $U'_\phi \subset \mathbb{RP}^n \times \mathbb{RP}^n$ which is defined as the set of all pairs of lines (A, B) in $\mathbb{R}^{n+1}$ such that $\phi(u, v) \neq 0$ for some $u \in A$ and $v \in B$. Now, all pairs of lines of the form (A, A) belong to U'_ϕ. Then, for $A \neq B$, the path from A to B is defined as above (rotating A toward B in the plane, spanned by A and B, in the positive direction determined by the orientation), and for $A = B$, we choose the constant path at A. Continuity is not violated.

A vector-valued nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$ determines k scalar maps $\phi_1, \dots, \phi_k : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ (the coordinates) and the above described neighborhoods U_{ϕ_i} cover the product $\mathbb{RP}^n \times \mathbb{RP}^n$ minus the diagonal. Since $n + 1 < k$, we may use Lemma 5.7. Hence, we may replace the initial nonsingular map by such an f that for any $u \in \mathbb{R}^{n+1}$, $u \neq 0$, the first coordinate $\phi_1(u, u)$ of $f(u, u)$ is positive. The open sets $U'_{\phi_1}, U_{\phi_2}, \dots, U_{\phi_k}$ cover $\mathbb{RP}^n \times \mathbb{RP}^n$. We have described explicit motion planning strategies over each of these sets. Therefore, $\mathsf{TC}(\mathbb{RP}^n) \leq k$. ■

Proposition 6.4. For $n = 1, 3, 7$, $\mathsf{TC}(\mathbb{RP}^n) = n + 1$. □

Proof. The upper bound $\mathsf{TC}(\mathbb{RP}^n) \leq n + 1$ follows by repeating the arguments of the proof of Proposition 6.3 applied to the nonsingular maps $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by Lemma 5.3. The lower bound $\mathsf{TC}(\mathbb{RP}^n) \geq n + 1$ is completely general, see (4.5). ■

Proof of Theorem 6.1. For $n \neq 1, 3, 7$, the claim follows by combining Propositions 6.2 and 6.3 with Lemma 5.4 and Corollary 4.4. The special cases $n = 1, 3, 7$ are covered by Proposition 6.4. ■

7 Motion planners and immersions $\mathbb{RP}^n \to \mathbb{R}^k$

In this section, we show that the problem of computing the topological complexity of the motion planning problem in $\mathbb{RP}^n$ is equivalent to the immersion problem for the real projective spaces.

Theorem 7.1. For any $n \neq 1, 3, 7$, the number $\mathbf{TC}(\mathbb{RP}^n)$ equals the smallest k such that the projective space $\mathbb{RP}^n$ admits an immersion into $\mathbb{R}^{k-1}$. □

The proof uses our main Theorem 6.1 and the following theorem of Adem, Gitler, and James proven in [2].

Theorem 7.2. There exists an immersion $\mathbb{RP}^n \to \mathbb{R}^k$ (where $k > n$) if and only if there exists an axial map $\mathbb{RP}^n \times \mathbb{RP}^n \to \mathbb{RP}^k$. □

Proof of Theorem 7.1. The proof is obtained by combining Theorem 6.1 with Lemma 5.6 and Theorem 7.2. ■

The next Theorem 7.3 is in fact a part of Theorem 7.1 stated above. However, in its proof, we give a direct construction, starting from an immersion $\mathbb{RP}^n \to \mathbb{R}^k$ of an open covering $\{U_i\}$ of $\mathbb{RP}^n \times \mathbb{RP}^n$ with continuous motion planning strategies s_i over each open set U_i.

Theorem 7.3. Suppose that the projective space $\mathbb{RP}^n$ can be immersed into $\mathbb{R}^k$. Then $\mathbf{TC}(\mathbb{RP}^n) \leq k + 1$. □

Proof. Let $\mathbb{RP}^n$ be immersed into $\mathbb{R}^k$. Fix a frame in $\mathbb{R}^k$ and extend it by parallel translation to a continuous field of frames. Projecting orthogonally onto $\mathbb{RP}^n$, we find k continuous tangent vector fields $v_1, v_2, \ldots, v_k$ on $\mathbb{RP}^n$ such that the vectors $v_i(p)$ (where $i = 1, 2, \ldots, k$) span the tangent space $T_p(\mathbb{RP}^n)$ for any $p \in \mathbb{RP}^n$.

Let $U_0 \subset \mathbb{RP}^n \times \mathbb{RP}^n$ denote the set of pairs of lines (A, B) in $\mathbb{R}^{n+1}$ making an acute angle.

A nonzero tangent vector v to the projective space $\mathbb{RP}^n$ at a point A (which we understand as a line in $\mathbb{R}^{n+1}$) determines a line $\widehat{v}$ in $\mathbb{R}^{n+1}$, which is orthogonal to A, that is, $\widehat{v} \perp A$. The vector v also determines an orientation of the two-dimensional plane spanned by the lines A and $\widehat{v}$.

For $i = 1, 2, \ldots, k$, let $U_i \subset \mathbb{RP}^n \times \mathbb{RP}^n$ denote the open set of all pairs of lines (A, B) in $\mathbb{R}^{n+1}$ such that the vector $v_i(A)$ is nonzero and the line B makes an acute angle with the line $\widehat{v_i(A)}$.

The sets $U_0, U_1, \ldots, U_k$ cover $\mathbb{RP}^n \times \mathbb{RP}^n$. Indeed, given a pair (A, B), there exist indices $1 \leq i_1 < \cdots < i_n \leq k$ such that the vectors $v_{i_r}(A)$, where $r = 1, \ldots, n$, span the tangent space $T_A(\mathbb{RP}^n)$. Then, the lines

$$A, \widehat{v_{i_1}(A)}, \ldots, \widehat{v_{i_n}(A)} \tag{7.1}$$

span the Euclidean space $\mathbb{R}^{n+1}$ and therefore the line B makes an acute angle with one of these lines. Hence, (A, B) belongs to one of the sets $U_0, U_{i_1}, \ldots, U_{i_n}$.

Now, we want to describe a continuous motion planning strategy over each set U_i, where $i = 0, 1, \ldots, k$. First, we define it over U_0. Given a pair $(A, B) \in U_0$, we rotate A towards B with constant velocity in the two-dimensional plane spanned by A and B so that A sweeps the acute angle. This clearly defines a continuous motion planning section $s_0 : U_0 \to P(\mathbb{RP}^n)$. Our continuous motion planning strategy $s_i : U_i \to P(\mathbb{RP}^n)$, where $i = 1, 2, \ldots, k$, is a composition of two motions. First we rotate line A toward the line $\widehat{v_i(A)}$ in the two-dimensional plane spanned by A and $\widehat{v_i(A)}$ in the direction determined by the orientation of this plane (see above). On the second step, we rotate the line $\widehat{v_i(A)}$ towards B along the acute angle similarly to the action of s_0. ■

The inverse statement of Theorem 6.1, allowing to construct an immersion $\mathbb{RP}^n \to \mathbb{R}^{k-1}$ starting from a motion planner for $\mathbb{RP}^n$, is not explicit; it is based on a long chain of constructions: Theorem 4.1, Corollary 4.4, Proposition 6.2, and then Theorem of Adem, Gitler, and James [2].

8 Some consequences of the main results

From Theorems 6.1 and 7.1, Lemma 5.4, and formula (4.5), we derive the following corollary.

Corollary 8.1. For all n distinct from $1, 3, 7$, one has $\mathbf{TC}(\mathbb{RP}^n) > n + 1$. The equality $\mathbf{TC}(\mathbb{RP}^n) = n + 1$ holds if and only if n equals $1, 3$, or 7. □

The second statement follows from Lemma 5.4 and Propositions 6.3 and 6.4 above.

Corollary 8.2. For any n, $\mathbf{TC}(\mathbb{RP}^n) \le 2n$. If n is a power of 2, then it is an equality, that is,

$$\mathbf{TC}\left(\mathbb{RP}^{2^{r-1}}\right) = 2^r. \tag{8.1}$$

☐

Proof. By the Whitney theorem, the projective space $\mathbb{RP}^n$ (for $n > 1$) admits an immersion into $\mathbb{R}^{2n-1}$, and the inequality follows (for $n \ne 1, 3, 7$) from Theorem 7.1. If n is $1, 3$, or 7, the inequality also holds since then $\mathbf{TC}(\mathbb{RP}^n) = n + 1 \le 2n$. Equality (8.1) is implied by Theorem 4.5. ■

Corollary 8.3. If $n \le n'$, then $\mathbf{TC}(\mathbb{RP}^n) \le \mathbf{TC}(\mathbb{RP}^{n'})$. ☐

Propositions 6.2 and 6.3 give the following corollary.

Corollary 8.4. The number $\mathbf{TC}(\mathbb{RP}^n)$ equals the Schwarz genus of the two-fold covering $S^n \times_{\mathbb{Z}_2} S^n \to \mathbb{RP}^n \times \mathbb{RP}^n$. It also coincides with the smallest k such that the vector bundle $k(\xi \otimes \xi)$ over $\mathbb{RP}^n \times \mathbb{RP}^n$ admits a nowhere zero section. ☐

Two statements of Corollary 8.4 are clearly equivalent, see the arguments in the proof of Corollary 4.4.

Milgram [14] constructed, for any odd n, a nonsingular map

$$\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{2n+1-\alpha(n)-k(n)}. \tag{8.2}$$

Here, $\alpha(n)$ denotes the number of ones in the dyadic expansion of n, and $k(n)$ is a nonnegative function depending only on the mod (8) residue class of n with $k(1) = 0$, $k(3) = k(5) = 1$, and $k(7) = 4$.

Applying Proposition 6.3, we obtain the following estimate.

Corollary 8.5. For any odd n,

$$\mathbf{TC}\left(\mathbb{RP}^n\right) \le 2n + 1 - \alpha(n) - k(n). \tag{8.3}$$

☐

Using the table on [9, page 273] and our Theorem 7.1, one finds the topological complexity of $\mathbb{RP}^n$ for all $n \le 23$, see Table 8.1.

Table 8.1

n	$\mathsf{TC}(\mathbb{RP}^n)$
1	2
2	4
3	4
4	8
5	8
6	8
7	8
8	16
9	16
10	17
11	17
12	19
13	23
14	23
15	23
16	32
17	32
18	33
19	33
20	35
21	39
22	39
23	39

9 Motion planners in $\mathbb{RP}^n$ with $n \leq 7$

We describe explicitly a motion planner in $\mathbb{RP}^2$ which may be used to solve the task of moving a line through the origin in $\mathbb{R}^3$.

We may view $\mathbb{R}^3$ as embedded into $\mathbb{R}^4$ (the set of quaternions) via the map $(x_1, x_2, x_3) \mapsto x_1 + x_2 i + x_3 j$, where $i, j, k \in \mathbb{R}^4$ are the imaginary units. Restricting the nonsingular map $\mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$ onto $\mathbb{R}^3 \subset \mathbb{R}^4$, we obtain a nonsingular map $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^4$. Explicitly, it is given by the formula

$$f(x, y) = \langle x, y \rangle - \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} i - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} j - \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} k, \tag{9.1}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, and $\langle x, y \rangle$ denotes the scalar product of x and y. Repeating the construction given in the proof of Proposition 6.3, we obtain four open subsets U_1, U_2, U_3, and U_4 of $\mathbb{RP}^2 \times \mathbb{RP}^2$, covering the product $\mathbb{RP}^2 \times \mathbb{RP}^2$. Each U_α, where $\alpha = 1, 2, 3, 4$, corresponds to the scalar pairing ϕ_α obtained from f by considering only the αth coordinate

$$f(x, y) = \phi_1(x, y) + \phi_2(x, y)i + \phi_3(x, y)j + \phi_4(x, y)k. \tag{9.2}$$

The set U_1 consists of the pairs of lines in $\mathbb{R}^3$ making an acute angle. The set U_2 consists of pairs of lines in $\mathbb{R}^3$ such that their projections onto the x_1x_2-plane span this plane. The sets U_3 and U_4 are defined similarly with the x_1x_2-plane replaced by the x_2x_3-plane and x_1x_3-plane, correspondingly.

Each functional ϕ_α defines a continuous motion planning strategy over the set U_α, see proof of Proposition 6.3. For example, the motion planning strategy over U_1 is obvious. If lines A and B make an acute angle, we rotate A towards B in the 2-plane spanned by A and B so that A sweeps the acute angle. If $A = B$, then A stays fixed.

A continuous motion planning strategy over the set U_2 can be described as follows. Let A and B be two lines in $\mathbb{R}^3$ so that their projections onto the x_1x_2-plane span this plane. Fix an orientation of the x_1x_2-plane. For any pair $(A, B) \in U_2$, we obtain an orientation of the 2-plane spanned by A and B. Then, we rotate A towards B in this 2-plane in the direction of the orientation.

Over the sets U_3 and U_4, we act similarly.

This example illustrates how one may use the nonsingular maps of Lemma 5.3 to explicitly construct motion planners in projective spaces $\mathbb{RP}^n$ with $n \leq 7$. We skip the details.

Notice that $\mathbb{RP}^3$ is homeomorphic to the Lie group SO(3), the configuration space of the rigid body, fixed at a point in 3-space. The topological complexity of this problem was computed earlier in [6, Section 8].

The immersion theory of projective spaces provides a rich variety of quite sophisticated nonsingular maps (see, e.g., [12, 14]) which maybe used to build, similarly to the above construction, explicit motion planners on higher-dimensional projective spaces $\mathbb{RP}^n$.

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