A proof of Culter's theorem on the existence of periodic orbits in polygonal outer billiards

Serge Tabachnikov*

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Abstract

We prove a recent theorem by C. Culter: every polygonal outer billiard in the affine plane has a periodic trajectory.

This note contains a proof of a recent theorem by Chris Culter; he obtained this result as an undergraduate participant of the 2004 Penn State REU program.[1] A complete account of Culter's work involving a more general class of maps and a more detailed analysis of their periodic orbits will appear in his paper, currently in progress.

An outer billiard table is a compact convex domain P. Pick a point x outside P. There are two support lines from x to P; choose one of them, say, the right one from the view-point of x, and reflect x in the support point. One obtains a new point, y, and the transformation $T : x \mapsto y$ is the outer (a.k.a. dual) billiard map, see figure 1. The map T is not defined if the support line has a segment in common with the outer billiard table. In this note, P is a convex n-gon; the set of points for which T or any of its iterations is not defined is contained in a countable union of lines and has zero measure. For ease of exposition, we assume that P has no parallel sides.

Outer billiards were introduced in [8] and popularized in [6, 7]; we refer to [1, 16, 17] for surveys. Here we are concerned with the existence of periodic

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*Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA; e-mail: `tabachni@math.psu.edu`

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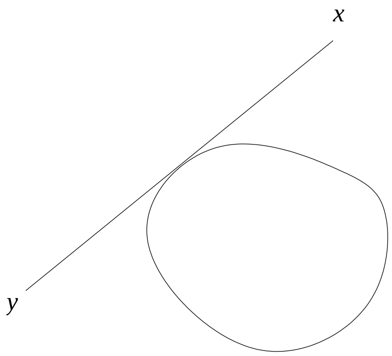

Figure 1: Definition of the outer billiard map

trajectories of the outer billiard map. For the conventional, inner, billiards it is an outstanding open problem whether every polygon has a periodic billiard path. The best result so far is a theorem of R. Schwartz: every obtuse triangle with the obtuse angle not greater than 100° has a periodic trajectory, see [10, 11, 12]. Note also that both inner and outer polygonal billiards on the sphere S^2 may have no periodic trajectories at all, see [4].

It will be convenient to consider the second iteration T^2 of the outer billiard map. Connecting the consecutive points of a periodic trajectory of T^2, one obtains a closed polygonal line. The number of turns made by this line about the billiard table is called the rotation number. The main result is as follows.

Theorem 1 *The map T^2 has a periodic trajectory that lies outside of any compact neighborhood of P and has rotation number 1.*

Proof of Theorem. For every outer billiard, not necessarily polygonal, the asymptotic dynamics of the map T^2 at infinity has the following description; see the sited surveys or [18, 19, 20]. A bird's eye view of a outer billiard is almost a point and the map T is almost the reflection in this point. More precisely, after rescaling, the distance between a point x and $T^2(x)$ is very small, and the evolution of a point under T^2 appears a continuous clockwise motion along a centrally symmetric curve R.

In our case, R is a convex $2n$-gon, and each vector $(x, T^2(x))$ belongs to a finite set $\{\pm v_1, \dots, \pm v_n\}$. These vectors are as follows. For every direction, other than the directions of the sides, there exists a pair of parallel support lines to P; the vector v_i is twice the vector connecting the respective support vertices of P, see figure 2. The vectors v_i are ordered according to the cyclic order of the pairs of support vertices of P. For example, if P is a triangle then the vectors v_i are twice the sides of P and R is an affine-regular hexagon.

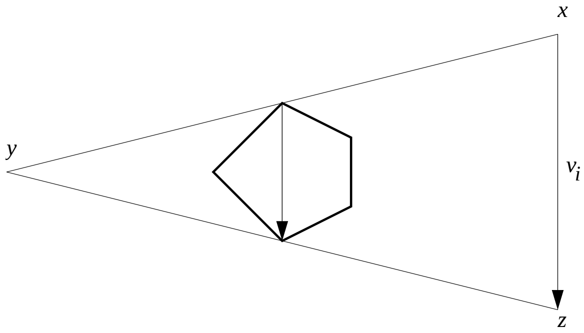

Figure 2: The second iteration of the outer billiard map

The polygon R is defined only up to dilation. Choose a representative R_0 of R (say, having unit area). To every side of R_0 there corresponds "time", the ratio of the length of this side to the magnitude of the respective vector v_i. One obtains a collection of "times" $(t_1, \ldots, t_k)$. If a different representative of R is chosen then the vector $(t_1, \ldots, t_k)$ is multiplied by a number, and hence the ratios t_i/t_j are well defined. The polygon P is called quasi-rational if all the numbers t_i are rational multiples of each other. For example, lattice polygons are quasi-rational and so are affine-regular ones. It is known that the orbits of the outer billiard about a quasi-rational polygon are bounded, see [3, 5, 9] or the cited surveys. Recently R. Schwartz proved that polygonal outer billiards about a special class of quadrilaterals have orbits escaping to infinity – see [13, 14] and the forthcoming book [15].

The map T is a piece-wise central symmetry. Since the composition of two central symmetries is a parallel translation, the map T^2 is a piece-wise parallel translation. Consider the map T^2 far away from the polygon P. Then the discontinuities of T^2 are $2n$ rays: the clockwise extensions of the sides of P and the reflections of these rays in the opposite vertices of P (a vertex opposite to a side is the one farthest from it). The lines containing these $2n$ rays form n strips $S_1, \ldots, S_n$ whose intersection contains P, see figure 3. The strips are numbered cyclically clockwise.

To summarize, one can enumerate the strips S_i and the vectors v_i consistently so that, far away from P, the map T^2 has the following description: starting at strip i, repeatedly add the vector v_i (or its negative, whichever points in the clockwise direction) to the point until the points lands in strip $i+1$, then repeatedly add the vector v_{i+i} until the point lands in strip $i+2$, etc.

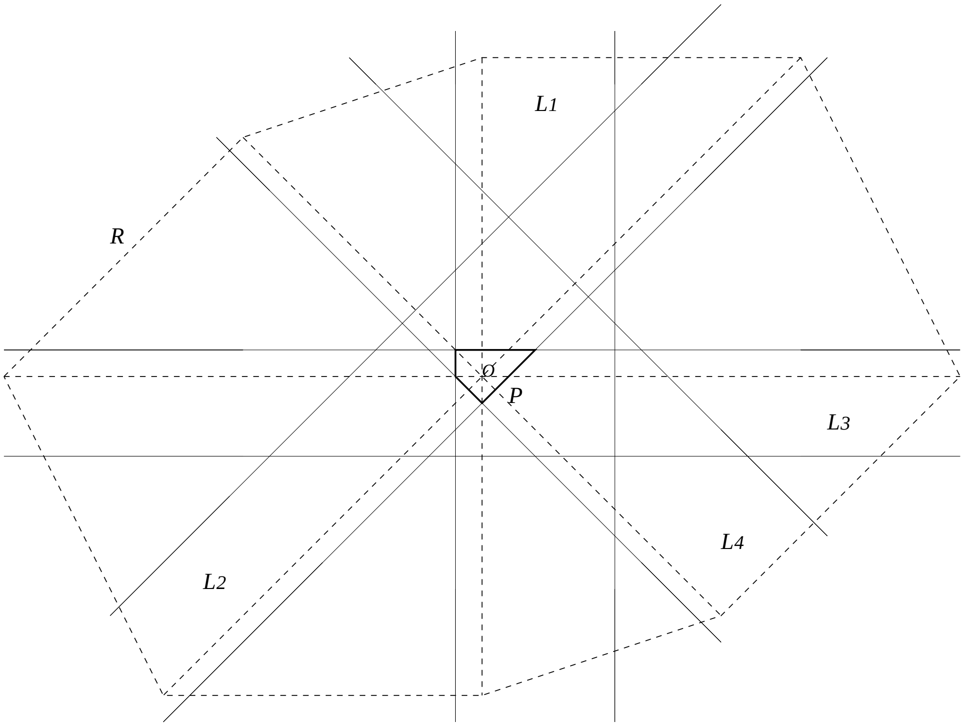

Figure 3: The lines L_i, the strips S_i and the polygon R

Choose an origin O inside P and consider the lines $L_1, \ldots, L_n$ through O parallel to the sides of P. Fix the above described polygon R so that O is its center. Denote by qR the dilation of R with coefficient q. These polygons can be constructed by choosing a starting point on L_1, drawing the line in the direction v_1 until its intersection with L_2, then drawing the line in the direction v_2 until its intersection with L_3, etc.

Let $p_1, \ldots, p_n$ be positive integers. Denote by $Q(p_1, \ldots, p_n)$ the centrally symmetric $2n$-gon whose sides are given by the vectors

$$p_1 v_1, p_2 v_2, \ldots, p_n v_n, -p_1 v_1, \ldots, -p_n v_n$$

and whose center is O. We wish to show that, for an appropriate choice of $p_1, \dots, p_n$, the polygon $Q(p_1, \dots, p_n)$ is an orbit of the map T^2. For this, the vertices of $Q(p_1, \dots, p_n)$ should lie inside the strips S_i (the opposite vertices in the same strip).

Clearly, there is $\varepsilon > 0$ (depending only on P and the choice of the origin) such that if the vertices of an $2n$-gon Q are ε-close to the respective vertices of a polygon qR then the vertices of Q lie inside the strips S_i. We claim that there exist arbitrarily large real q and integers $p_1, \dots, p_n$ such that the respective vertices of qR and $Q(p_1, \dots, p_n)$ are within ε from each other.

For the claim to hold, it will suffice to have

$$|qt_i - p_i| < \delta, \quad i = 1, \dots, n \tag{1}$$

where $\delta > 0$ is a small enough constant. Indeed, the first vertex of the polygon qR is

$$-\frac{1}{2}\sum_1^n qt_i v_i,$$

whereas that of the polygon $Q(p_1, \dots, p_n)$ is

$$-\frac{1}{2}\sum_1^n p_i v_i,$$

and similarly for the other vertices.

Finally, consider the torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$, and let F_t be the constant flow with the vector $(t_1, \dots, t_n)$. Then (1) means that $F_q(O)$ is δ-close to O where $O = (0, \dots, 0)$. Indeed, the flow F_t is either periodic, and then $F_q(O) = O$ for q forming an arithmetic progression, or quasi-periodic and thus returning arbitrarily close to the initial point infinitely often. $\Box$

Remarks. 1. A composition of a number of central symmetries is either a central symmetry or a parallel translation. It follows that a k-periodic point of the outer billiard map about a polygon has a polygonal neighborhood consisting of periodic points with period k or $2k$ (the latter holds if k is odd).

2. The choice of the origin O in the proof was arbitrary and did not play a role in the argument. A nearby choice of the origin leads to the same result

because of the stability of the periodic orbits in the sense of the previous remark.

3. The density of the numbers q satisfying (1) is positive. One can deduce that the lower density of the set of periodic trajectories described in Theorem 1 is also positive.

4. A periodic trajectory of the polygonal outer billiard map is called stable if, under an arbitrary small perturbation of the outer billiard polygon P, the trajectory is also perturbed but not destroyed. A criterion for stability is known, see [16]. Enumerate the vertices of P counterclockwise as $A_1, \dots, A_n$. An even-periodic orbit of the dual billiard map is encoded by the sequence of vertices in which the consecutive reflections occur. One obtains a cyclic word W in the letters $A_1, \dots, A_n$. The orbit is stable if and only if each appearance of every letter in an odd position in W is balanced by its appearance in an even position. By this criterion, the periodic trajectories of Theorem 1 are stable.

5. Theorem 1 implies the existence of uniformly large periodic basins consisting of convex polygons that are permuted by the dynamics. The complement of this periodic set consists of points with infinite orbits (and, of course, the points at which some iteration of the outer billiard map is not defined). The situation was completely analyzed in the case of the affine-regular pentagon in [18, 19]: the set of points with infinite orbits is the closure of the discontinuity set and is a fractal whose Hausdorff dimension is equal to $\ln 6/\ln(\sqrt{5}+2)$. Computer experiments (see [16, 18, 19]) suggest that the situation is similar for other affine-regular n-gons (except for $n = 3, 4, 6$ which are lattice polygons). Conjecturally, this description extends to all polygons.

6. The outer billiard map T can be defined in the hyperbolic plane as well. In this case, the map T extends to the circle at infinity as a continuous map. This circle map has the rotation number ρ. If the the outer billiard n-gon is sufficiently large then all orbits of T escape to infinity, see [2]; however, in the case of "large" n-gons, $\rho = 1/n$, and hence the outer billiard map at infinity has an n-periodic orbit. Likewise, if ρ is rational then there is a periodic trajectory at infinity. We conjecture that, for every outer billiard polygon in the hyperbolic plane, the outer billiard map has periodic orbits, even if $\rho \notin \mathbf{Q}$.

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