References

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On Functions with Zero Mean over a Finite Group*

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1. The following observation was made by Arnold [1]. Let f be a real trigonometric polynomial of degree n, with zero constant term, that changes sign at exactly two points. These points divide the circle into two arcs.

Theorem 1. *The ratio of the lengths of these arcs is not less than* $1/n$. *If equality holds, then* f *is unique up to a factor and a rotation of the circle. The roots of this extremal polynomial form a regular* $(n+1)$*-gon, and the multiplicities of all roots are equal to* 2, *except for two neighboring roots.*

An estimate close to the first assertion of this theorem is contained in [2]. In this note we present a few variations on this theme.

We start with a simple general observation. Let M be a finite measure space and let $f\colon M \to \mathbb{R}$ be a measurable function. Denote by $m(f)$ the relative measure of the set $\{x \in M \mid f(x) \le 0\}$. Let G be a finite group of measure-preserving transformations of M; denote by $\bar f$ the average over G:

$$\bar f(x) = \frac{1}{|G|} \sum_{g \in G} f(gx).$$

Theorem 2. *If* $\bar f \equiv 0$, *then* $m(f) \ge 1/|G|$.

The proof follows from the fact that every G-orbit intersects the set $\{x \mid f(x) \le 0\}$.

2. For the case in which M is $\mathbb{S}^1$ with Lebesgue measure and G is the cyclotomic group of rotations of order n, we obtain the following corollary.

Theorem 3. *If the Fourier expansion of a function* f *contains no harmonics whose order is a multiple of* n *(including the constant term), then* $m(f) \ge 1/n$. *In particular,* $m(f) \ge 1/n$ *for any trigonometric polynomial, of degree less than* n, *with zero mean.*

Consider the case in which equality is attained. Let f be a trigonometric polynomial of degree N without harmonics whose order is a multiple of n, and let $m(f) = 1/n$. Denote by k the number of components of the set $\{x \mid f(x) < 0\}$.

Theorem 4. $k \le N/(n-1)$.

In particular, if $N = n-1$, then f changes the sign exactly twice, and therefore coincides with an extremal polynomial from Theorem 1.

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Proof. Consider the intervals $I_1, \dots, I_k$ that compose the set $\{x \mid f(x) < 0\}$, together with their rotations about the angles of $2\pi j/n$, $j = 0, \dots, n-1$. These nk intervals I_{ij} cover $\mathbb{S}^1$ (except for their endpoints) and do not overlap. Choose $i \in \{1, \dots, k\}$, and put unit masses at the left endpoints of the intervals I_{ij}, $j = 0, \dots, n-1$. The function f is orthogonal to the resulting $\mathbb{Z}_n$-invariant measure, and $f \geq 0$ at any point of the support of this measure (because none of the above points belongs to an interval of the form $I_1, \dots, I_k$). Therefore, each of these n points is a root of f. At most two of these points are boundary for the intervals I_l (otherwise some intervals I_{lj} would intersect). Thus, $n-2$ of these roots are multiple, and the total multiplicity of these n roots is at least $2(n-1)$. As i varies from 1 to k, we obtain $2(n-1)k$ roots (counted according to their multiplicities). A trigonometric polynomial of degree N has at most $2N$ roots, and this proves the theorem.

Recall that a Chebyshev system of order $2n-1$ is a $(2n-1)$-dimensional space of functions on the circle such that any nonzero function from this space has at most $2(n-1)$ roots, counted according to their multiplicities. The assertion of Theorem 3 on the trigonometric polynomials of degree less than n remains valid for functions from $\mathbb{Z}_n$-invariant Chebyshev systems of order $2n-1$. This result of [2] is a consequence of our Theorem 2 and of the following lemma from [2]. Since the proof in [2] is quite long, we give a simple argument here.

Lemma 5. *Let f be an element of a $\mathbb{Z}_n$-invariant Chebyshev system of order $2n-1$ on the circle. If the mean value of f is zero, then its average over $\mathbb{Z}_n$ is zero.*

Proof. The mean value of f is equal to that of $\bar{f}$ (and is zero). Hence, $\bar{f}$ has roots. Since $\bar{f}$ is $\mathbb{Z}_n$-invariant, $\bar{f}$ has at least $2n$ roots counted according to their multiplicities. At the same time $\bar{f}$ belongs to a Chebyshev system of order $2n-1$, and therefore $\bar{f} \equiv 0$.

3. We give three analogs of Theorem 3 for functions of several variables.

(i) Let $P(z_1, \dots, z_k)$ be a complex polynomial of degree $n-1$ with zero constant term, and let $f = \operatorname{Re} P$. We regard f as a function on the unit sphere $\mathbb{S}^{2k-1} \subset \mathbb{C}^k$ (with standard measure).

Theorem 6. $m(f) \geq 1/n$.

Indeed, the average of f over the group $\mathbb{Z}_n$ generated by the multiplication of any complex coordinate by $\exp(2\pi i/n)$ is the zero function.

(ii) Let $f\colon \mathbb{T}^k \to \mathbb{R}$ be a trigonometric polynomial, with zero constant term, whose degrees with respect to the variables are $n_1 - 1, \dots, n_k - 1$.

Theorem 7. $m(f) \geq 1/(n_1 \cdots n_k)$.

Here $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ acts by rotating the factors about the angles of $2\pi/n_1$, $2\pi/n_2$, etc., and $\bar{f} \equiv 0$.

(iii) Assume that for every subset $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, a polynomial $g_I(x_{i_1}, \dots, x_{i_k})$ skew-symmetric with respect to the group S_k and a polynomial $h_I(x_1, \dots, x_n)$ symmetric with respect to the group S_n are chosen. Let f be the restriction of the polynomial $\sum_{I \subset \{1,\dots,n\}} g_I h_I$ to the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

Theorem 8. $m(f) \geq 1/n!$.

Indeed, the average of f over the symmetric group S_n acting on $\mathbb{R}^n$ by coordinate permutations is the zero function.

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