

REMARKS ON MAGNETIC FLOWS AND MAGNETIC BILLIARDS, FINSLER METRICS AND A MAGNETIC ANALOG OF HILBERT'S FOURTH PROBLEM

SERGE TABACHNIKOV

ABSTRACT. We interpret magnetic billiards as Finsler ones and describe an analog of the string construction for magnetic billiards. Finsler billiards for which the law “angle of incidence equals angle of reflection” are described. We characterize the Finsler metrics in the plane whose geodesics are circles of a fixed radius. This is a magnetic analog of Hilbert’s fourth problem asking to describe the Finsler metrics whose geodesics are straight lines.

1. INTRODUCTION AND BACKGROUND MATERIAL

This paper concerns the motion of a charged particle in a magnetic field, a popular object of study in mathematics and mathematical physics. In the Euclidean plane, the strength of the magnetic field is given by a function $B(x_1, x_2)$, and the particle moves with constant speed, satisfying the equation

$$(1) \quad \ddot{x} = B(x_1, x_2)J\dot{x} \quad \text{where} \quad J(v_1, v_2) = (-v_2, v_1).$$

If one fixes the speed $|v|$ then the magnetic field prescribes the curvature of the trajectory at every point. In particular, if the field is constant then the trajectories are circles of radius $|v|/|B|$, the Larmor radius. Our sign convention is that if $B > 0$ then the circles are traversed in the counter-clockwise direction.

In general, a magnetic field on a Riemannian manifold M is a closed differential 2-form β , and the magnetic flow is the Hamiltonian flow of the Riemannian Hamiltonian function $|p|^2/2$ on the cotangent bundle T^*M with respect to the twisted symplectic structure $\omega + \pi^*(\beta)$ where $\omega = dp \wedge dq$ is the standard symplectic structure on T^*M and $\pi: T^*M \rightarrow M$ is the projection. We refer to [4, 13, 14, 16, 17, 20, 21] for a variety of results on magnetic flows on Riemannian manifolds.

If a charged particle is confined to a domain with ideally reflecting boundary then one has a magnetic billiard. The particle moves inside according to equation (1) and undergoes elastic reflections off the boundary: the tangential component of the velocity remains the same and the normal one changes sign. This amounts to the familiar law of geometric optics: the angle of incidence equals that of reflection. Magnetic billiards have attracted considerable attention: see [10, 11, 12, 18, 28, 29, 30, 35]; see also [25] for a survey of various aspects of billiard systems.

In this paper we interpret a magnetic flow as a geodesic flow of a Finsler metric. We mostly consider the 2-dimensional case. In Section 2 we interpret magnetic billiards as Finsler ones and describe the magnetic version of the string construction that recovers a billiard table by a caustic of the billiard map. We also characterize the Finsler metrics for which the Finsler billiard enjoys the familiar law “angle of incidence equals angle of reflection” (Theorem 2 and Corollary 3). In Section 3 we describe the Finsler metrics in the plane whose geodesics are circles of a fixed radius; we give analytic and synthetic descriptions in Theorems 6,8,9. This is an analog of the celebrated fourth Hilbert problem of describing the Finsler metrics in a domain in projective space whose geodesics

are straight lines [1, 3, 15, 22]. Our solution has an unexpected connection to another classical and well studied question: the Pompeiu problem [32, 33, 34].

We will now review basics of Finsler geometry (see, e.g., [2, 6, 8, 23]) and Finsler billiards, recently introduced in [19]. Finsler geometry describes the propagation of light in an inhomogeneous anisotropic medium. This means that the velocity of light depends on the point and the direction. There are two equivalent descriptions of this process corresponding to the Lagrangian and the Hamiltonian approaches in classical mechanics, and we will mostly use the former.

The optical properties of a medium are described by a quadratically convex smooth hypersurface, called the indicatrix, in the tangent space at each point. The indicatrix consists of the velocity vectors of the propagation of light at a point in all directions. It plays the role of the unit sphere in Riemannian geometry. We do not assume that the indicatrices are centrally symmetric.

Equivalently, a Finsler metric on a manifold M is determined by a smooth nonnegative fiberwise convex Lagrangian function $L(x, v)$ on the tangent bundle TM , homogeneous of degree 1 in the velocity:

$$L(x, tv) = tL(x, v) \quad \text{for all } t > 0.$$

The restriction of L to a tangent space $T_x M$ gives the Finsler length of vectors in $T_x M$, and the indicatrix at x is the unit level hypersurface of $L(x, v)$. Given a smooth curve $\gamma : [a, b] \rightarrow M$, its length is

$$\mathcal{L}(\gamma) = \int_a^b L(\gamma(t), \gamma'(t)) dt.$$

The integral does not depend on the parameterization. A Finsler geodesic is an extremal of the functional \mathcal{L} . The Finsler geodesic flow is the flow in TM in which the foot point of a vector moves along the Finsler geodesic tangent to it, so that the vector remains tangent to this geodesic and preserves its Finsler length. The Finsler geodesic flow is described by the Euler-Lagrange equation

$$(2) \quad dL_v(x, v)/dt - L_x(x, v) = 0 \quad \text{or} \quad L_{vv}\dot{v} + L_{vx}v - L_x = 0.$$

The dual, Hamiltonian approach describes the propagation of light in terms of wave fronts and the Finsler geodesic flow as a Hamiltonian flow in the cotangent bundle T^*M . Let $I \subset T_x M$ be the indicatrix. The figuratrix $J \subset T_x^* M$ is the dual hypersurface constructed as follows. Given a vector $u \in I$, the respective covector $p \in J$ is defined by the conditions:

$$\text{Ker } p = T_u I \quad \text{and} \quad p(u) = 1.$$

This gives a diffeomorphism $I \rightarrow J$, called the Legendre transform. In the same way as the field of indicatrices determines the Lagrangian L , the field of figuratrices determines a Hamiltonian H on T^*M . The Hamiltonian vector field of the function H is also called the Finsler geodesic flow; the Legendre transform identifies the two flows.

Example 1: Hilbert's fourth problem in dimension two. The Euclidean metric is given by the Lagrangian $L(x, v) = |v|$; its geodesics are straight lines. Such metrics are called projective. Following [1], let us describe all symmetric projective Finsler metrics in the plane, that is, a solution to Hilbert's fourth problem in dimension 2.

A synthetic approach, due to Busemann, makes use of integral geometry, namely, the Crofton formula [24]. Consider the set of oriented lines in the plane, topologically, the cylinder. An oriented line can be characterized by its direction $\alpha \in [0, 2\pi)$ and its signed distance p from the origin. The 2-form $\omega_0 = dp \wedge d\alpha$ is the standard area form on the space of oriented lines; this symplectic form is a particular case of a symplectic structure on the space of trajectories of a Hamiltonian system on a fixed energy level, in particular, the space of oriented geodesics of a Finsler metric – see, e.g., [7]

and the discussion in Section 3. The Crofton formula gives the Euclidean length of a plane curve γ in terms of ω_0 . The curve determines a function on the space of oriented lines, the number of intersections of a line l with γ . Then

$$(3) \quad \text{length}(\gamma) = (1/4) \int \#(l \cap \gamma) \omega_0.$$

Let $f(p, \alpha)$ be a positive continuous function. Then $\omega = f(p, \alpha) dp \wedge d\alpha$ is also an area form on the space of oriented lines. Formula (3), with ω replacing ω_0 , defines a projective Finsler metric, and all such metrics can be obtained by an appropriate choice of the function f .

Next we describe an analytic solution to Hilbert's fourth problem in dimension two. First, a Lagrangian $L(x, v)$, homogeneous of degree 1 in v , gives a projective Finsler metric if and only if the mixed second partial derivative matrix L_{xv} is symmetric; this is Hamel's theorem of 1903, and it holds in any dimension. The Lagrangians satisfying Hamel's condition have the following integral representation:

$$(4) \quad L(x_1, x_2, v_1, v_2) = \int_0^{2\pi} |v_1 \cos \phi + v_2 \sin \phi| f(x_1 \cos \phi + x_2 \sin \phi, \phi) d\phi$$

where $f(p, \phi)$ is a smooth positive function on the cylinder representing the space of oriented lines. Moreover, if f is even in ϕ then it is uniquely determined by L . The function f is the same as in (3): the length of a curve with respect to the Finsler metric (4) is given, up to a multiplicative constant, by (3). If f depends on the angle ϕ only then one obtains a translation invariant metric, called a Minkowski metric. If f is a constant then one has the Euclidean metric.

We mention in passing that nonsymmetric projective Finsler metrics received much less attention; a class of examples is provided by nonsymmetric Minkowski metrics.

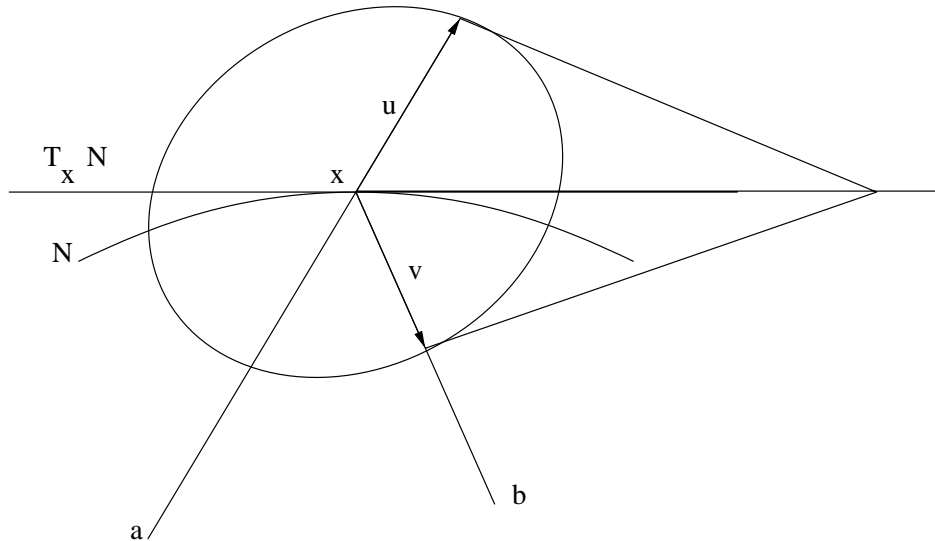


FIGURE 1. Finsler billiard reflection

Let M be a 2-dimensional Finsler manifold with boundary, a curve N . The Finsler billiard system is defined in [19] as follows. A point moves inside M freely, according to the Finsler geodesic flow, until it hits the boundary. The reflection is described in terms of the indicatrix I at the impact point x – see Figure 1. The vectors u and v are the Finsler unit vectors along the incoming and outgoing trajectories. The tangent lines to I at u and v are concurrent with the tangent line to N at x . This definition satisfies a variational principle: for every points $a, b \in M$, the reflection point x extremizes the Finsler length $|ax| + |xb|$. If the indicatrix is a circle centered at the origin then the vectors u and v make equal angles with the boundary curve N ; this is the familiar law of Euclidean billiard reflection. The multi-dimensional version of the Finsler billiard reflection is defined similarly, and we do not dwell on it – see [19].

Example 2: projective Finsler billiard reflection. Consider a symmetric projective Finsler metric (4). In polar coordinates, $v_1 = r \cos \alpha$, $v_2 = r \sin \alpha$, and

$$(5) \quad L(x_1, x_2, r, \alpha)/2 = r \int_{\alpha-\pi/2}^{\alpha+\pi/2} \cos(\alpha - \phi) f(x_1 \cos \phi + x_2 \sin \phi, \phi) d\phi.$$

Let α be the direction of the billiard curve at the impact point x and β and γ the directions of the incoming and the outgoing billiard trajectories – see Figure 2. The projective Finsler reflection law specializes to the following formula.

Lemma 1.1. *One has:*

$$\int_{\gamma-\pi/2}^{\beta-\pi/2} \cos(\alpha - \phi) f(x_1 \cos \phi + x_2 \sin \phi, \phi) d\phi = \int_{\gamma+\pi/2}^{\beta+\pi/2} \cos(\alpha - \phi) f(x_1 \cos \phi + x_2 \sin \phi, \phi) d\phi.$$

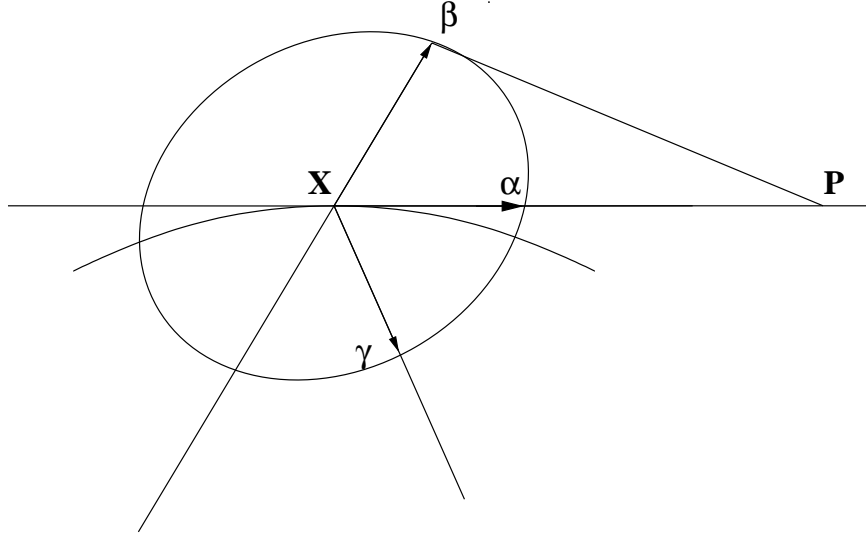


FIGURE 2. Deriving the projective Finsler billiard reflection law

For example, if $f = 1$ then integration yields: $\cos(\beta - \alpha) = \cos(\gamma - \alpha)$ or $\beta - \alpha = \gamma - \alpha$, the familiar law of equal angles.

Proof. Denote the integral in (5) by $g(x, \alpha)$. Then the polar equation of the indicatrix at point x , chosen as the origin, is $r = 1/g(x, \alpha)$. It is a matter of a straightforward calculation to find the coordinates of the intersection point P in Figure 2:

$$P(\beta) = \frac{(\cos \alpha, \sin \alpha)}{g(\beta) \cos(\alpha - \beta) + g'(\beta) \sin(\alpha - \beta)}.$$

Equating $P(\beta)$ and $P(\gamma)$ yields:

$$(6) \quad g(\beta) \cos(\alpha - \beta) + g'(\beta) \sin(\alpha - \beta) = g(\gamma) \cos(\alpha - \gamma) + g'(\gamma) \sin(\alpha - \gamma).$$

It follows from (5) that

$$g'(\beta) = - \int_{\beta-\pi/2}^{\beta+\pi/2} \sin(\beta - \phi) f(x_1 \cos \phi + x_2 \sin \phi, \phi) d\phi,$$

and similarly for γ . It remains to substitute into (6) and to collect terms. \square

2. MAGNETIC BILLIARDS AS FINSLER BILLIARDS

Consider the plane motion of a charged particle in a magnetic field with strength $B(x_1, x_2)$. The Lagrangian for this motion is

$$\bar{L}(x, v) = \frac{1}{2}|v|^2 + \alpha(x)(v)$$

where $\alpha(x) = f(x_1, x_2) dx_1 + g(x_1, x_2) dx_2$ is a differential 1-form such that $d\alpha = -B(x_1, x_2) dx_1 \wedge dx_2$. The choice of α is not unique: one can always add a closed 1-form to a Lagrangian without effecting the dynamics. The Euler-Lagrange equation for \bar{L} is (1). In particular, the Lagrangian for a constant magnetic field is

$$\bar{L}(x, v) = \frac{1}{2}|v|^2 + \frac{B}{2}[v, x]$$

where $[\ , \]$ is the cross-product.

Following the Maupertuis principle (see, e.g., [6]), we replace the Lagrangian \bar{L} by

$$(7) \quad L(x, v) = |v| + \alpha(x)(v).$$

The extremals of the Lagrangian (7) coincide with those of \bar{L} , corresponding to motion with unit speed. In particular, the extremals of

$$(8) \quad L(x, v) = |v| + \frac{1}{2R}[v, x]$$

are the counterclockwise oriented circles of radius R .

The Lagrangian (7) defines a nonsymmetric Finsler metric in the domain where $L(x, v) > 0$ for all $v \neq 0$. This is the case if $|\alpha(x)| < 1$, and we assume this condition to hold throughout this section. In other words, we assume that the magnetic field is sufficiently weak. Under this assumption, we consider the unit speed magnetic flow as the Finsler geodesic flow.

Consider a plane domain and the magnetic billiard inside it. One also has the Finsler billiard inside the domain, associated with the Lagrangian (7). One expects the two systems to coincide, that is, to have the same reflection laws.

Theorem 1. *The Finsler billiard reflection law, associated with the Lagrangian (7), is the law of equal angles: the angle of incidence equals the angle of reflection.*

Proof. The indicatrix of the Finsler metric at point x is given by the equation $|v| + \alpha(x)(v) = 1$. Choose Cartesian coordinates in such a way that $\alpha(x)(v) = tv_1$ for some $t \in \mathbf{R}$. Then the equation of the indicatrix can be rewritten as

$$(9) \quad (1 - t^2)^2 \left(v_1 + \frac{t}{1 - t^2} \right)^2 + (1 - t^2)v_2^2 = 1.$$

Recall that the equation of a conic, with focus located at the origin in the (v_1, v_2) -plane, is

$$\frac{(v_1 + c)^2}{a^2} + \frac{v_2^2}{b^2} = 1$$

where $a^2 - c^2 = b^2$. Clearly, (9) has this form. Hence the indicatrix is an ellipse, with focus located at the origin.

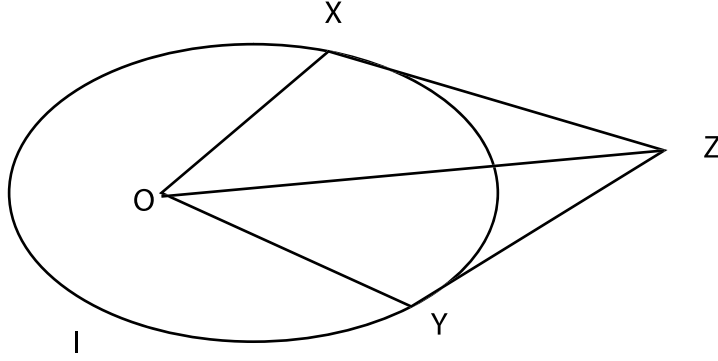


FIGURE 3. Poncelet's first little theorem

Thus the theorem reduces to the following geometrical property of conics. Let I be a conic with focus O , let $X, Y \in I$, and let Z be the intersection point of the tangent lines to I at X and Y . Then the line OZ bisects the angle XOY – see Figure 3. This property holds indeed; it is known as the Poncelet “first little theorem”, see [9]. This completes the proof. \square

In fact, the law of equal angles is characteristic of the conics, centered at a focus.

Theorem 2. *Let I be a smooth plane curve, star-shaped with respect to point O , with the following property. Let $X, Y \in I$ be arbitrary points, Z be the intersection point of the tangent lines to I at X and Y ; then the line OZ bisects the angle XOY . It follows that I is a conic with focus O .*

Proof. Let O be the origin, and give I a parameterization $I(t)$ so that $[I(t), I'(t)] \equiv 1$; here $[\ , \]$ denotes the determinant made by two vectors. Then $I''(t) = -f(t)I(t)$ for some function $f(t)$; thus we view I as an orbit in a central force field. We claim that $f(t) = C/|I(t)|^3$ for a constant C . Assuming this claim, it follows that I is an orbit in Newton's force field, and therefore a conic with focus O .

Let $X = I(t_1) = I_1, Y = I(t_2) = I_2$. A direct computation yields the point Z :

$$Z = I_1 + \frac{[I_2 - I_1, I_2']}{[I_1', I_2']} I_1' = I_2 + \frac{[I_1 - I_2, I_1']}{[I_2', I_1']} I_2'.$$

The equal angle condition reads: $[Y, Z]/|Y| = [Z, X]/|X|$, or

$$(10) \quad |I_2| (1 - [I_1, I_2']) = |I_1| (1 - [I_2, I_1']).$$

Now set: $t_1 = t$, $t_2 = t + \varepsilon$ and shorthand $I(t)$ to I and $f(t)$ to f . Then the Taylor expansion yields:

$$I_2 = I \left(1 - \frac{\varepsilon^2}{2} f - \frac{\varepsilon^3}{6} f' \right) + I' \left(\varepsilon - \frac{\varepsilon^3}{6} f \right) + O(\varepsilon^4),$$

$$I_2' = I' \left(1 - \frac{\varepsilon^2}{2} f - \frac{\varepsilon^3}{3} f' \right) - I \left(\varepsilon f + \frac{\varepsilon^2}{2} f' + \frac{\varepsilon^3}{6} (f'' - f^2) \right) + O(\varepsilon^4),$$

and

$$|I_2| = |I| + \varepsilon \frac{I \cdot I'}{|I|} + O(\varepsilon^2).$$

Substitute into (10) and collect terms to obtain $|I|^2 f' + 3I \cdot I' f = 0$. This differential equation is easily solved: $f'/f = -3I \cdot I'/|I|^2$ and hence $f = C/|I|^3$, as claimed. \square

As a consequence, we obtain a description of the Finsler metrics for which the Finsler billiard reflection law is the law of equal angles.

Corollary 3. *The Finsler billiard reflection satisfies the law “angle of incidence equals angle of reflection” for every billiard curve if and only if the metric is given by a Lagrangian*

$$(11) \quad L(x, v) = f(x)(|v| + \alpha(x)(v))$$

where $f(x)$ is a non-vanishing function and $\alpha(x)$ is a 1-form.

Proof. Replacing a metric by a conformally-equivalent one changes the indicatrices by a dilation and does not affect the law of equal angles. Theorem 1 implies that the metrics (11) satisfy this law of equal angles. Conversely, if this law holds then, by Theorem 2, the indicatrices are ellipses (depending on the point of the plane), centered at their foci. A general equation of such an ellipse is $f(x)(|v| + \alpha(x)(v)) = 1$, and the result follows. \square

It is shown in [19] that some familiar properties of the usual billiards extend to the Finsler ones. Although [19] concerns symmetric Finsler metrics, the results hold in the nonsymmetric case as well; however one should be careful with the order of points: the distance from A to B may differ from the distance from B to A . Let us consider the case of a constant magnetic field, that is, the Finsler metric given by the Lagrangian (8) whose geodesics are counterclockwise oriented arcs of radius R .

Let A and B be two points on an arc of radius R with center C and angle measure θ . Denote the Finsler distance between points by $d(A, B)$ and identify points with their position vectors. Let $\mathcal{L}(\gamma)$ denote the Finsler length of a curve γ .

Lemma 2.1. *One has:*

$$d(A, B) = \frac{1}{2}\theta R + \frac{1}{2R}[B - A, C].$$

For a simple oriented closed curve γ , one has:

$$(12) \quad \mathcal{L}(\gamma) = l(\gamma) - \frac{1}{R}S(\gamma)$$

where $l(\gamma)$ and $S(\gamma)$ are the Euclidean length and the Euclidean signed area bounded by γ .

Proof. To obtain (12), one integrates $|v| + [v, x]/(2R)$ over γ and makes use of the fact that $[v, x]/2$ is negative the derivative of the signed area swept out by the position vector of γ .

Let O be the origin. The distance $d(A, B)$ equals the integral of $|v| + [v, x]/(2R)$ over the arc AB . The integral of $|v|$ is the arclength of the arc, that is, θR . The integral of $[v, x]/(2R)$ equals $-S/R$ where S is the area of the curvilinear triangle OAB .

Assume first that $O = C$. Then the latter area is $\theta R^2/2$, and $d(A, B) = \theta R/2$. If the origin is translated through vector C then the Lagrangian changes by the term $[v, C]/(2R)$, and its integral by $[B - A, C]/(2R)$. This yields the first formula. \square

Remark 2.2. Formula (12), along with its proof, holds for closed immersed curves as well: the area term should be understood as the integral of the 1-form $(xdy - ydx)/2$ over the curve.

Moreover, we will need to extend the definition of area to a class of curves with singularities called wave fronts.

A contact element in the plane is a 1-dimensional subspace of the tangent space at a point (the foot point of the contact element). The space of co-oriented contact elements of the plane is the spherization of the cotangent bundle $T^*\mathbf{R}^2$, topologically, a solid torus. The space of contact elements has a natural contact structure defined by the “skating” condition: the contact plane is spanned by infinitesimal motions of a contact element such that the foot point stays in the contact line.

A smooth co-oriented plane curve γ lifts to the space of co-oriented contact elements by assigning the co-oriented tangent line to every point of γ . The lifted curve $\tilde{\gamma}$ is Legendrian, i.e., it is everywhere tangent to the contact structure. Conversely, given an oriented Legendrian curve $\tilde{\gamma}$ in the space of co-oriented contact elements, its projection $p(\tilde{\gamma})$ to the plane is called a wave front. Generically, wave fronts have semi-cubic cusps. The area bounded by a co-oriented wave front is defined as the integral of the 1-form $p^*(xdy - ydx)/2$ over the Legendrian curve $\tilde{\gamma}$.

Let γ be a smooth oriented curve. The orientation of γ determines a coorientation: the pair (coorientation vector, orientation vector) gives the positive orientation of the plane. If γ is a counterclockwise oriented simple curve then the positive coorientation is the outward one. Given a real number t , consider the parallel curve $\Gamma(t)$ at distance t from γ . The curve $\Gamma(t)$ is the time- t wave front, starting at $\gamma = \Gamma(0)$. More precisely, one translates the contact elements of γ in the orthogonal direction through distance t (along the coorienting vector, if $t > 0$, and in the opposite direction, if $t < 0$), and the obtained 1-parameter family of contact elements constitutes a Legendrian curve in the space of co-oriented contact elements. The curve $\Gamma(t)$ is the respective wave front. If γ is a positively oriented circle of radius r then $\Gamma(t)$ is a circle of radius $r + t$.

Formula (12) admits the following interpretation.

Lemma 2.3. *For an oriented simple closed curve γ , one has:*

$$(13) \quad \mathcal{L}(\gamma) = \frac{1}{R} (\pi R^2 - S(\Gamma(-R)))$$

where the area $S(\Gamma(-R))$ is understood as explained after Remark 2.2.

Proof. For a closed immersed curve, the following well-known formula holds:

$$S(\Gamma(t)) = S(\gamma) + t\ell(\gamma) + \pi t^2 w$$

where w is the Whitney winding number of γ ; this formula extends to wave front as well. Thus the right hand side of (12) equals $(1/R)(\pi R^2 - S(\Gamma(-R)))$, and the result follows from Lemma 2.1. \square

Remark 2.4. Formula (13) expresses Finsler lengths in terms of areas, and it serves a magnetic analog of the Crofton formula (3). The curve $\Gamma(-R)$ is the locus of the centers of positively oriented circles of radius R , tangent to γ and having the same orientation as γ at the tangency point. The curve γ can be reconstructed from $\Gamma(-R)$ as the envelope of the family of circles of radius R centered at points of $\Gamma(-R)$.

Formula (13) also resembles the area-length duality for spherical curves, discussed in [5, 26, 27]. Let γ be a simple smooth closed curve on the unit sphere, and let Γ be its spherically dual curve, namely, the curve $\Gamma(\pi/2)$, in the sense of spherical geometry. Then one has: $l(\gamma) = 2\pi - S(\Gamma)$.

Let us return to billiards. Recall that a caustic of a 2-dimensional billiard is a curve Γ inside it with the following property: if a segment of a billiard trajectory is tangent to Γ then so is the reflected segment. Given a convex caustic Γ , can one reconstruct the billiard table? For the usual, Euclidean billiard the answer is given by the string construction: a billiard curve N is the locus of points X of a string of fixed length, wrapped around Γ – [9, 25].

It is shown in [19] that the string construction extends to Finsler billiards as well. For nonsymmetric Finsler metrics one needs to consider oriented caustics Γ so that the orientation of billiard trajectories, tangent to Γ , agrees with the orientation of Γ . Applying these considerations to billiards in a constant magnetic field, we obtain the following corollary.

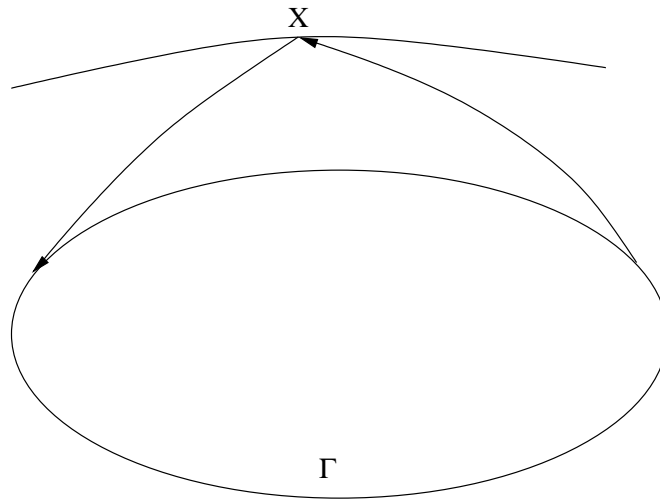


FIGURE 4. Magnetic string construction

Let Γ be an oriented closed convex curve. For a point X outside of Γ , let $F(X)$ be the Finsler length, associated with the metric (8), of the shortest closed curve from X to X around the obstacle Γ , whose orientation agrees with that of Γ – see Figure 4.

Corollary 4. *The level curves of the function $F(X)$ are the boundaries of magnetic billiard tables that have Γ as a caustic.*

Recall the optical property of an ellipse: a ray emanating from one focus reflects to another focus. As a particular case of Corollary 4, one may construct a magnetic analog of an ellipse.

Corollary 5. *Let A and B be fixed points (“foci”) and N be the locus of points X such that $d(A, X) + d(X, B) = \text{const}$. Then every trajectory of the magnetic billiard, starting at A , reflects in N to B .*

Note that the two foci play different roles and cannot be interchanged in the above formulation. If the points A and B merge then the “ellipse” N becomes a Euclidean circle centered at this point.

3. FINSLER METRICS WHOSE GEODESICS ARE CIRCLES OF A FIXED RADIUS

In this section we develop a magnetic analog of the solution to Hilbert's fourth problem, outlined in Example 1 above. Start with an analytic description of the Lagrangians, homogeneous of degree 1 in the velocity, whose extremals are positively oriented circles of radius R .

Lemma 3.1. *The extremals of $L(x, v)$ are positively oriented circles of radius R if and only if L satisfies the equation:*

$$(14) \quad \frac{|v|}{R} L_{vv}(Jv) + L_{vx}(v) = L_x$$

where $J(v_1, v_2) = (-v_2, v_1)$.

Proof. Let $x(t)$ be a parameterized curve, $v = x'$. The curve is a counterclockwise oriented circle if and only if

$$\left(\frac{v}{|v|} \right)' = \frac{1}{R} J(v).$$

Differentiate to express the acceleration vector:

$$(15) \quad v' = \frac{|v|}{R} J(v) + \frac{(v \cdot v')v}{|v|^2}.$$

Since the Lagrangian is homogeneous of degree 1, the Euler equation $L_v v = L$ holds, and hence $L_{vv}(v) = 0$. It remains to substitute v' from (15) to the Euler-Lagrange equation

$$L_{vv}(v') + L_{vx}(v) = L_x,$$

and the result follows. \square

We are ready to prove the main analytical result of this section.

Theorem 6. *Every Lagrangian, homogeneous of degree 1 in the velocity, whose extremals are positively oriented circles of radius R can be represented, in polar coordinates, as follows:*

$$(16) \quad L(x, v) = L(x_1, x_2, r, \alpha) = r \left(\int_0^{\alpha+\pi/2} \cos(\alpha - \phi) g(x_1 + R \cos \phi, x_2 + R \sin \phi) d\phi + a(x_1, x_2) \cos \alpha + b(x_1, x_2) \sin \alpha \right)$$

where g is a positive density function in the plane such that the center of mass of every circle of radius R is its center, and a, b are two functions, satisfying

$$(17) \quad a_{x_2}(x_1, x_2) - b_{x_1}(x_1, x_2) = \frac{1}{R} g(x_1 + R, x_2).$$

Proof. In polar coordinates, $v_1 = r \cos \alpha$, $v_2 = r \sin \alpha$, and one has: $L(x, v) = |v| p(x_1, x_2, \alpha)$ for some function p .

Fix a point $x = (x_1, x_2)$ and consider the indicatrix I at x , chosen as the origin. The polar equation of I is $r = 1/p(x, \alpha)$. Therefore $p(x, \alpha)$ is the support function of the dual curve, the figuratrix J (see [24]). Parameterize J by the angle ϕ made by its tangent vector with the horizontal axis. Let $f(x, \phi)$ be the radius of curvature at point $J(\phi)$ and let $(a(x), b(x))$ be the coordinates of the point $J(0)$. One has:

$$J'(\phi) = f(x, \phi)(\cos \phi, \sin \phi),$$

and hence

$$J(\alpha + \pi/2) = J(0) + \int_0^{\alpha+\pi/2} f(x, \phi)(\cos \phi, \sin \phi) d\phi,$$

see Figure 5. It follows that

$$(18) \quad p(x, \alpha) = (\cos \alpha, \sin \alpha) \cdot J(\alpha + \pi/2) = a(x) \cos \alpha + b(x) \sin \alpha + \int_0^{\alpha+\pi/2} \cos(\alpha - \phi) f(x, \phi) d\phi.$$

Differentiating (18) twice, one recovers the function f from p :

$$f(x, \alpha + \pi/2) = p(x, \alpha) + p''(x, \alpha).$$

Note that every function of the form $p + p''$ is L^2 orthogonal to $\cos \alpha$ and $\sin \alpha$. Thus

$$(19) \quad \int_0^{2\pi} f(x, \alpha) \cos \alpha d\alpha = \int_0^{2\pi} f(x, \alpha) \sin \alpha d\alpha = 0;$$

this also follows from the integral representation (18) and periodicity of p as a function of α .

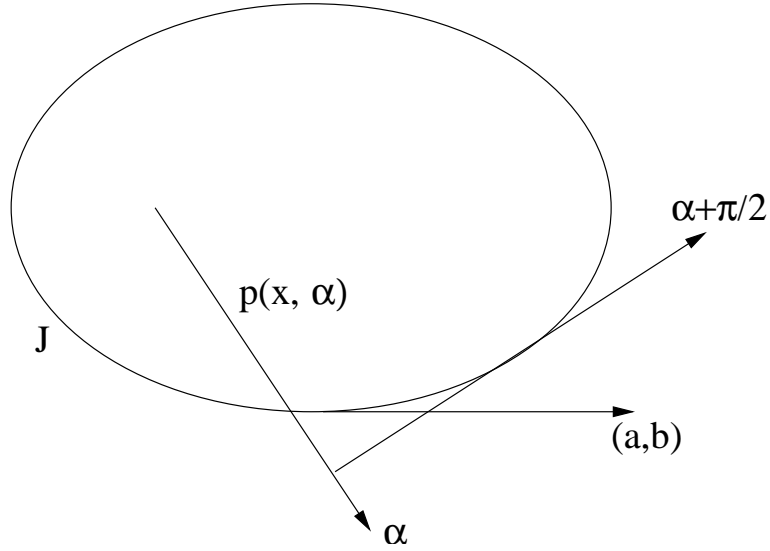


FIGURE 5. Integral representation of the support function

Next we use the equation (14) in the integral representation (18). One rewrites the differential operators $\partial^2 v$, $\partial v \partial x$ and ∂x in polar coordinates and applies them to $L(x_1, x_2, r, \alpha) = rp(x_1, x_2, \alpha)$, given by (18). Taking into account that $v = r(\cos \alpha, \sin \alpha)$, a computation reveals that

$$\frac{|v|}{R} L_{vv}(Jv) = \frac{rf(x, \alpha + \pi/2)}{R} (-\sin \alpha, \cos \alpha),$$

and

$$L_x - L_{vx}(v) = r \left(\int_0^{\alpha+\pi/2} (\cos \phi f_{x_2}(x, \phi) - \sin \phi f_{x_1}(x, \phi)) d\phi + a_{x_2}(x) - b_{x_1}(x) \right) (-\sin \alpha, \cos \alpha).$$

Therefore, by (14),

$$(20) \quad \frac{1}{R} f(x, \alpha + \pi/2) = \int_0^{\alpha + \pi/2} (\cos \phi f_{x_2}(x, \phi) - \sin \phi f_{x_1}(x, \phi)) d\phi + a_{x_2}(x) - b_{x_1}(x).$$

In particular,

$$(21) \quad \frac{1}{R} f(x, 0) = a_{x_2}(x) - b_{x_1}(x).$$

Differentiating (20) with respect to α , one gets:

$$(22) \quad \frac{1}{R} f_\alpha(x, \alpha) = \cos \alpha f_{x_2}(x, \alpha) - \sin \alpha f_{x_1}(x, \alpha).$$

We claim that

$$(23) \quad f(x_1, x_2, \alpha) = g(x_1 + R \cos \alpha, x_2 + R \sin \alpha)$$

for an appropriate function of two variables g .

Indeed, consider the vector field

$$\eta = \frac{1}{R} \partial \alpha + \sin \alpha \partial x_1 - \cos \alpha \partial x_2$$

on the solid torus $\mathbf{R}^2 \times S^1$. Then (22) can be written as $\eta(f) = 0$. The trajectories of η are:

$$\alpha(t) = \frac{t}{R}, \quad x_1(t) = -R \cos \left(\frac{t}{R} \right) + a, \quad x_2(t) = -R \sin \left(\frac{t}{R} \right) + b$$

where t is the “time” parameter and a, b are constants. One can take the plane $\alpha = 0$ as a section. Then the η -invariant function f is determined by its values on this section, a function of two variables g . Consider a point (α, x_1, x_2) . The trajectory through this point intersects the section at the point $(a, b) = (x_1 + R \cos \alpha, x_2 + R \sin \alpha)$. Hence $f(x_1, x_2, \alpha) = g(a, b)$, and (23) follows.

Equations (19) imply that the center of mass of the circle of radius R , with the density function g , centered at (x_1, x_2) , is the point (x_1, x_2) . Combining (21) and (23), we obtain (17). \square

Remark 3.2. The term $r(a(x) \cos \alpha + b(x) \sin \alpha)$ in (16), the formulation of Theorem 6, can be written as $\nu(x)(v)$ where $\nu(x) = a(x) dx_1 + b(x) dx_2$ is a 1-form. The choice of this form is not unique but $d\nu$ is uniquely determined by the function g via (17). This is consistent with the remark we already made: adding a closed 1-form to the Lagrangian does not affect the Euler-Lagrange equations.

As a corollary of Theorem 6 let us describe those metrics $L(x, v)$ given by (16) whose odd part $L(x, v) - L(x, -v)$ is a differential 1-form in v , that is, depends on the velocity linearly. Of course, the standard metric (8) has this property.

Corollary 7. *If $L(x, v) - L(x, -v)$ is a differential 1-form in v then $L(x, v)$ equals the standard metric (8), up to a constant multiplier and up to summation with an exact 1-form.*

Proof. Using the notation from Theorem 6, one has:

$$F(\alpha) := L(x, v) - L(x, -v) = r \left(\int_0^{\alpha + \pi/2} \cos(\alpha - \phi) g(x_1 + R \cos \phi, x_2 + R \sin \phi) d\phi + \right.$$

$$\int_0^{\alpha-\pi/2} \cos(\alpha-\phi) g(x_1 + R \cos \phi, x_2 + R \sin \phi) d\phi + 2(a(x_1, x_2) \cos \alpha + b(x_1, x_2) \sin \alpha).$$

The function $L(x, v) - L(x, -v)$ is linear in v if and only if $F(\alpha)$ is a linear combination of $\cos \alpha$ and $\sin \alpha$, that is, if $F(\alpha) + F''(\alpha) = 0$. A computation similar to that in the proof of Theorem 6 yields:

$$F(\alpha) + F''(\alpha) = g(x_1 - R \sin \alpha, x_2 + R \cos \alpha) - g(x_1 + R \sin \alpha, x_2 - R \cos \alpha).$$

Thus

$$g(x_1 - R \sin \alpha, x_2 + R \cos \alpha) = g(x_1 + R \sin \alpha, x_2 - R \cos \alpha)$$

for all x_1, x_2 and α , that is, the function g assumes equal values at every pair of points at distance $2R$. Therefore g is constant, and the result follows from (16). \square

To proceed, we recall basic facts about the symplectic reduction. Let (M, Ω) be a symplectic manifold and $H: M \rightarrow \mathbf{R}$ a Hamiltonian function. Consider the Hamiltonian vector field $\xi = \text{sgrad } H$. Since H is ξ -invariant, the field ξ is tangent to the level hypersurfaces of H . Consider such a hypersurface S , and assume that the space of trajectories of ξ on S is a smooth manifold $N = S/\xi$; locally, this is always the case. The restriction of Ω to S has 1-dimensional kernel spanned by ξ , and hence $\Omega|_S$ descends to a symplectic structure ω on N . This is the symplectic reduction of Ω .

One applies this construction as follows. Given a Finsler manifold M , the symplectic manifold in question is the cotangent bundle T^*M with its standard symplectic structure $dp \wedge dx$ where $x \in M$ is the position and $p \in T_x^*M$ the momentum. The function H is the Finsler metric Hamiltonian, and the hypersurface S consists of the unit covectors; it is fibered over M and the fibers are the figuratrices. The vector field ξ is the Finsler geodesic flow, and the space of trajectories identifies with the space of unparameterized oriented geodesics.

Consider the tangent bundle TM and the unit vector hypersurface U in it. The Legendre transform $(x, v) \mapsto (x, p = L_v)$ identifies U with S and the Finsler geodesic flow ζ on U with the geodesic flow ξ on S . The pull-back of the Liouville form pdx is the 1-form $\lambda = L_v dx$ on TM . The form λ is a contact form on U , and ζ is its Reeb vector field: $\lambda(\zeta) = 1$, $i_\zeta d\lambda = 0$. The reduction of the 2-form $d\lambda$ yields the symplectic structure on the quotient space U/ζ , the space of oriented Finsler geodesics.

Given a smooth curve γ on M , one lifts it to the curve $\tilde{\gamma}$ on U by assigning the unit tangent vector to every point of γ . Then the Finsler length of γ equals

$$\int_{\tilde{\gamma}} \lambda.$$

For a reference to this symplectic approach, see, e.g., [6, 7].

Now we are in a position to compute the symplectic structure on the space of circles of radius R , associated with the Lagrangian (16). A circle is characterized by its center, and the space of circles is the plane with Cartesian coordinates (u, v) .

Theorem 8. *The symplectic structure ω on the space of circles of radius R , associated with the Lagrangian (16), is given by the formula:*

$$(24) \quad \omega = -\frac{1}{R} g(u, v) du \wedge dv.$$

Proof. The manifold U consists of the Finsler unit tangent vectors in the plane and has coordinates α, x_1, x_2 . We use the notation from the proof of Theorem 6. The formulas derived in that proof yield:

$$\zeta = \frac{1}{p(x, \alpha)} \left(\cos \alpha \partial x_1 + \sin \alpha \partial x_2 + \frac{1}{R} \partial \alpha \right),$$

$$\lambda = L_v dx = \left(\int_0^{\alpha+\pi/2} \cos \phi f(x, \phi) d\phi + a(x) \right) dx_1 + \left(\int_0^{\alpha+\pi/2} \sin \phi f(x, \phi) d\phi + b(x) \right) dx_2$$

and, taking (20) and (21) into account,

$$d\lambda = f(x, \alpha + \pi/2) \left(\cos \alpha d\alpha \wedge dx_2 - \sin \alpha d\alpha \wedge dx_1 - \frac{1}{R} dx_1 \wedge dx_2 \right).$$

In view of (23),

$$(25)d\lambda = g(x_1 - R \sin \alpha, x_2 + R \cos \alpha) \left(\cos \alpha d\alpha \wedge dx_2 - \sin \alpha d\alpha \wedge dx_1 - \frac{1}{R} dx_1 \wedge dx_2 \right).$$

Now consider the projection $U \rightarrow U/\zeta = \mathbf{R}^2$. To compute the symplectic structure ω in \mathbf{R}^2 , consider a section $j: \mathbf{R}^2 \rightarrow U$ and let ω be the pull-back of $d\lambda$; the result is independent of the choice of j . As a section one may take

$$j(u, v) = (\alpha, x_1, x_2) \text{ with } \alpha = \frac{\pi}{2}, x_1 = u + R, x_2 = v,$$

see Figure 6. It remains to substitute into (25), and the result follows. \square

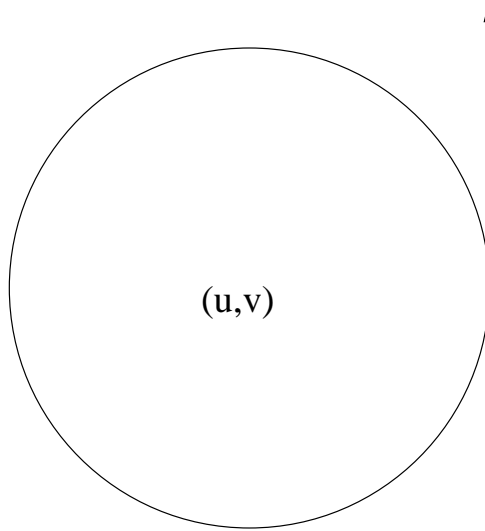


FIGURE 6. Computing the area form on the space of circles

As a consequence, the Finsler metric (16) can be recovered, up to summation with a closed 1-form, from the area form (24) – see Remark 3.2.

Next we consider an analog of formula (13) for a general Finsler metric (16) whose geodesics are circles of radius R . The following result expresses the Finsler length in terms of the area form on

the space of circles and is analogous to the synthetic solution to Hilbert's fourth problem, that is, the Crofton formula, used as a definition of a projective metric.

Theorem 9. *Given an oriented simple closed curve γ , one has:*

$$(26) \quad \mathcal{L}(\gamma) = S(\Gamma(-R)) + C$$

where $S(\Gamma(-R))$ is the area bounded by the curve $\Gamma(-R)$ with respect to the area form (24), and C is the common Finsler length of all positively oriented circles of radius R .

Proof. Note first that the geodesics are extremals of the length functional \mathcal{L} . The space of geodesics identifies with the plane which is a critical manifold of \mathcal{L} . A function is constant on its critical manifold, hence all positively oriented circles of radius R have equal Finsler length.

To prove the result we consider a variation of the curve γ and show that both sides of (26) have the same variations. This being established, one can deform γ to a circle of radius R for which the result holds.

Assume that γ is parameterized by the Euclidean arc-length. Then $\gamma' = (\cos \alpha, \sin \alpha)$. The lift of γ to U is the curve $\tilde{\gamma} = (\gamma, \alpha)$. Consider a variation of the curve, that is, a vector field w along γ . It is straightforward to compute that the respective variation of $\tilde{\gamma}$ is the vector field

$$(27) \quad \tilde{w} = w + [\gamma', v'] \partial \alpha.$$

One has:

$$\mathcal{L}(\gamma) = \int_{\tilde{\gamma}} \lambda$$

where λ is the contact form as in the proof of Theorem 8. Therefore the variation of the length $\mathcal{L}(\gamma)$ is given by the formula

$$\int_{\tilde{\gamma}} i_{\tilde{w}} d\lambda$$

where $d\lambda$ is as in (25). Let $k(t)$ be the curvature at $\gamma(t)$. Then $d\alpha = k dt$. A computation using (25) and (27) reveals that

$$(28) \quad \int_{\tilde{\gamma}} i_{\tilde{w}} d\lambda = \int g(x_1 - R \sin \alpha, x_2 + R \cos \alpha) [\gamma', w] \left(\frac{1}{R} - k \right) dt.$$

On the other hand, one has:

$$\Gamma(-R) = (X_1, X_2) = (x_1 - R \sin \alpha, x_2 + R \cos \alpha).$$

Therefore the variation of $\Gamma(-R)$ is given by the vector field

$$u = (w_1 - R[\gamma', v'] \cos \alpha, w_2 - R[\gamma', v'] \sin \alpha).$$

Then

$$dX_1 = (1 - Rk) \cos \alpha dt, \quad dX_2 = (1 - Rk) \sin \alpha dt.$$

Since $\omega = (-1/R) g(X_1, X_2) dX_1 \wedge dX_2$, it is straightforward to compute the variation of the area $S(\Gamma(-R))$:

$$\int_{\Gamma(-R)} i_u \omega = \int g(x_1 - R \sin \alpha, x_2 + R \cos \alpha) [\gamma', w] \left(\frac{1}{R} - k \right) dt.$$

This is the same as (28), and we are done. \square

Note the following corollary of formula (26).

Corollary 10. *The integral of the area form (24) is the same over all discs of radius R .*

Proof. Let γ degenerate to a point in (26), so that $\mathcal{L}(\gamma) = 0$. Then $\Gamma(-R)$ is a circle of radius R , and its ω -area equals $-C$. \square

Remark 3.3. One can give a somewhat different proof of Theorem 9 that does not use the specifics of the Euclidean plane and applies to other surfaces, for example, the sphere. Let us outline the argument. Pick a point O inside γ and consider an infinitesimally small loop δ around O whose orientation is the same as that of γ . The Finsler unit tangent vector fields to γ and to δ extend to a unit vector field in the annulus A bounded by γ and δ . This vector field provides a lift \tilde{A} of the annulus to U , and $\partial\tilde{A} = \tilde{\gamma} - \tilde{\delta}$. By Stokes' theorem,

$$\int_{\tilde{\gamma}} \lambda - \int_{\tilde{\delta}} \lambda = \int_{\tilde{A}} d\lambda.$$

The second integral on the left hand side is infinitesimally small. The integral on the right can be understood as the symplectic area of the set of circles of radius R whose centers lie between the curves $\Gamma(-R)$ and the circle of radius R , centered at O , and (26) follows. This also shows that the symplectic area of a circle of radius R is independent of its choice.

One can reverse the arguments and define the respective Finsler metric as in Theorem 6, starting with an area form $\omega = g(x_1, x_2) dx_1 \wedge dx_2$, satisfying the property that the ω -area of every disc of radius R is the same. Then the function $g(x_1, x_2)$ should be L^2 orthogonal to cosine and sine on every circle of radius R .

Lemma 3.4. *The integrals of a function g over all discs of radius R are the same if and only if g is orthogonal to cosine and sine on every circle of radius R .*

Proof. Let S be a circle of radius R with center $x = (x_1, x_2)$. Consider its variation given by an infinitesimal parallel translation through vector $v = (v_1, v_2)$. The variation of the ω -area of the disc is

$$\int_S i_v \omega = \int_0^{2\pi} g(x_1 + R \cos \alpha, x_2 + R \sin \alpha) (v_1 \sin \alpha - v_2 \cos \alpha) d\alpha.$$

This vanishes for all v if and only if g is orthogonal to $\cos \alpha$ and $\sin \alpha$. \square

How restrictive are these two equivalent conditions on function g ? This question goes to the heart of the Pompeiu problem, see [32, 33, 34]. Given a compact set K , one considers the continuous functions with zero integrals over all isometric images of K . For which sets K must such functions be identically zero? D. Pompeiu, who posed this problem in the late 1920s, erroneously thought that the disc in the plane has this property. In fact, if K is a disc then there are plenty of functions with zero integrals over all congruent discs; although there is a wealth of results on this subject, the general solution to the Pompeiu problem is not known yet.

If a function g has a constant integral over all discs of radius R then it can be written as $\text{Const} + h(x_1, x_2)$ where h has zero integrals over all discs of radius R . The following result, standard in the literature on the Pompeiu problem, provides a substantial supply of such functions. We need to recall the definition of the Bessel functions.

The Bessel functions $J_n(w)$, $n \in \mathbf{Z}$, are defined by the generating function

$$(29) \quad \exp\left(\frac{w}{2}\left(t - \frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(w) t^n.$$

An explicit formula is as follows:

$$J_n(w) = \sum_{j=0}^{\infty} \frac{(-1)^j w^{n+2j}}{2^{n+2j} (n+j)! j!}, \quad n \geq 0; \quad J_{-n}(w) = J_n(-w).$$

We will need the following property:

$$(30) \quad \int J_0(w) w dw = w J_1(w).$$

Lemma 3.5. *Let a be a root of the first Bessel function J_1 , and let $f(\beta)$ be a function on the circle. Then the functions*

$$(31) \quad h(x_1, x_2) = \int_0^{2\pi} \cos\left(\frac{a}{R}(x_1 \cos \beta + x_2 \sin \beta)\right) f(\beta) d\beta$$

and

$$h(x_1, x_2) = \int_0^{2\pi} \sin\left(\frac{a}{R}(x_1 \cos \beta + x_2 \sin \beta)\right) f(\beta) d\beta$$

have zero integrals over all discs of radius R .

One may also take linear combinations of such functions over different roots of J_1 .

Proof. Let D be the disc of radius R centered at the origin, and let ξ be its characteristic function. The condition on function h reads: $\xi * h = 0$ where $*$ denotes the convolution. Take the Fourier transform to obtain:

$$(32) \quad \hat{\xi} \hat{h} = 0.$$

Let us compute $\hat{\xi}$:

$$\hat{\xi}(\lambda) = \int_D e^{-i\lambda x} dx = \int_0^R \int_0^{2\pi} e^{-ir\rho \cos(\alpha-\beta)} d\alpha r dr$$

where $x = r(\cos \alpha, \sin \alpha)$, $\lambda = \rho(\cos \beta, \sin \beta)$. One has:

$$-ir\rho \cos(\alpha - \beta) = \frac{r\rho}{2} (e^{i\theta} - e^{-i\theta})$$

where $\theta = \alpha - \beta - \pi/2$. Using the definition of Bessel functions (29), it follows that

$$\int_0^{2\pi} e^{-ir\rho \cos(\alpha-\beta)} d\alpha = \sum_n J_n(r\rho) \int_0^{2\pi} e^{in\theta} d\theta = 2\pi J_0(r\rho).$$

By (30), one has:

$$\int_0^R J_0(r\rho) r dr = \frac{R}{\rho} J_1(R\rho)$$

and hence

$$\hat{\xi}(\lambda) = \frac{2\pi R}{\rho} J_1(R|\lambda|).$$

The condition (32) holds if the support of \hat{h} is contained in the union of circles, centered at the origin, whose radii are a/R where a is a root of J_1 . Fix one such root and let

$$\hat{h}(\rho, \beta) = f(\beta) \delta_{\frac{a}{R}}(\rho).$$

Taking the inverse Fourier transform yields:

$$h(x_1, x_2) = \int_0^{2\pi} \int_0^\infty f(\beta) \delta_{\frac{a}{R}}(\rho) e^{i\rho(x_1 \cos \beta + x_2 \sin \beta)} \rho d\rho d\beta = \\ \frac{a}{R} \int_0^{2\pi} f(\beta) e^{i\frac{a}{R}(x_1 \cos \beta + x_2 \sin \beta)} d\beta.$$

One concludes by taking real and imaginary parts. \square

For example, let f in (31) be the delta function δ_0 . Then $h = \cos(ax_1/R)$. Substitute $g = 1 + \cos(ax_1/R)$ into (16) to obtain an “exotic” Finsler metric whose geodesics are circles of radius R .

Remark 3.6. One may consider the problem of description of the Finsler metrics whose geodesics are circles of a fixed geodesic radius R on the unit sphere. In this case one does not have a Lagrangian function as in (7) because the area form on the sphere is not exact. Nevertheless one can determine the length of closed curves via integral geometry: Theorem 9 and Corollary 10 still apply, see Remark 3.3. However the situation is different on S^2 , as far as the continuous functions are concerned whose integrals vanish over all geodesic discs of radius R . For all values of R , except a countable dense set, such functions are identically zero, see [31]. This implies an interesting “almost everywhere” rigidity: for a generic R , there is only one, standard, metric whose geodesics are circles of radius R ; this unique metric is an analog of the metric (8) in \mathbf{R}^2 . Of course, in the plane, all values of the radius R are equivalent, due to similarity.

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DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY PARK, PA 16802, USA

E-mail address: tabachni@math.psu.edu

URL: www.math.psu.edu/tabachni