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# INVARIANTS OF SMOOTH TRIPLE POINT FREE PLANE CURVES

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## 0. Introduction

A generic closed immersed plane curve has only transversal double points. In a generic one-parameter family of such curves two singular events may happen: a simple self-tangency and a triple point at which every two branches are transversal (see Fig. 1).

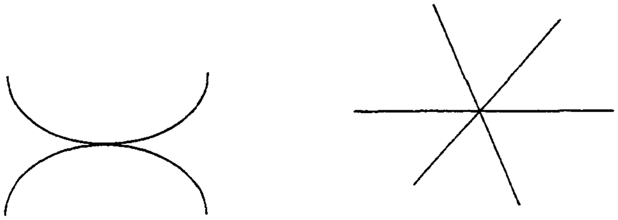


Fig. 1

One considers immersed curves, free from one, or both, of these singularities, modulo isotopies. The theories of such objects are similar to smooth knot theory, whose subject is immersed curves in 3-space free from selfintersections.

This work concerns curves without triple points (we do not specify, so far, whether the curves have one or several components, or whether they are oriented). Thus immersed curves are considered up to diffeomorphisms of the plane and the local moves, whose oriented and nonoriented versions are shown in Fig. 2 (as usual, the curves outside the shown fragments are understood to be identical). These moves will be refered to as the self-tangency moves.

A remarkable contribution to the theory of plane curves was made in a recent series of papers by V. Arnold [1,2,3]. The point of view is that of the singularity

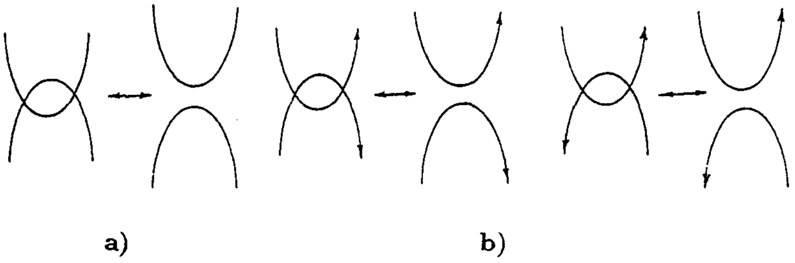


Fig. 2

theory, introduced into knot theory by V. Vassiliev [10,4]. Namely, one considers the infinite dimensional space of immersed plane curves and distinguishes in it a codimension-one singular hypersurface, called the discriminant, consisting (in our case) of curves with triple points. The equivalence classes of curves correspond to path components of the complement of the discriminant, and numerical invariants of equivalence classes are locally constant functions in this complement.

To define an invariant (modulo an additive constant) one needs to specify its jumps under crossing the discriminant at its smooth points. These jumps must be consistent in the sense that the total increment of the invariant along a loop in the space of curves vanishes.

This technique was used by Vassiliev in his recent study of *plane ornaments* [11]; an ornament is a collection of continuous closed curves, free from triple intersections of three distinct components. This approach is also applicable to *doodles* [5,7]; a doodle is a collection of continuous closed curves, free from any triple intersections.

We describe the simplest, and most basic, invariant of triple point free curves due to Arnold. Consider closed one-component oriented immersed curves in the plane. Let a generic one-parameter family of such curves be given in which a triple point surgery occurs — see Fig. 3 (we will call it the triple point move).

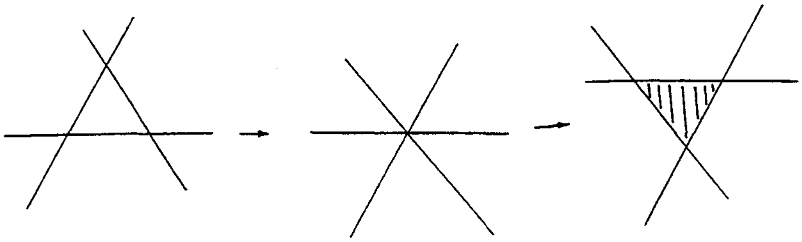
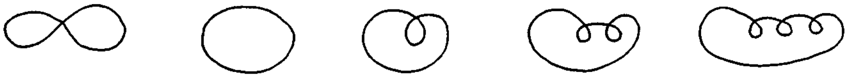


Fig. 3

Assign a sign to a triple point. Consider a small newly-born shaded triangle. Its sides are oriented because so is the curve. Choose a point on the curve outside this triangle. As one traverses the curve, starting at the chosen point, the sides of the triangle are visited in a certain cyclic order. This order does not depend on the choice of the starting point. Thus the sides of the triangle get cyclically oriented. Let $q \in \{0, 1, 2, 3\}$ be the number of sides whose intrinsic orientation agrees with the cyclic one. Define the sign of the triple point to be $(-1)^q$. This sign is independent of the orientation of the curve.

The triple point move is a transversal crossing of the discriminant by a path in the space of immersed curves. The above choice of sign gives the discriminant a coorientation.

The space of immersed closed plane curves is disconnected. According to the classical Whitney theorem [13] the only invariant of the regular homotopy, in which both self-tangency and triple point moves are allowed, is the *rotation number* $r$, i.e. the number of complete turns made by a tangent vector to the curve (if the curve is oriented, counterclockwise turns count as positive, and clockwise as negative). The canonical representatives of the regular homotopy equivalence classes are shown in Fig. 4.



| r | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| St | 0 | 0 | 1 | 2 | 3 |

Fig. 4

The integer-valued invariant $St$ ("strangeness") of closed immersed curves, free from triple points, is defined axiomatically: it increases by 1 under a positive triple point move and is normalized on the cannonical curves as shown in Fig. 4. This normalization makes $St$ additive under the connected summation of immersions. $St$ is an invariant of the first order (and the only local first order invariant): its jumps are the same under any positive triple point move.

A new contribution to Arnold's theory of plane curves has been made in the recent preprints by O. Viro [12] and A. Shumakovich [9]. In particular, these papers contain new combinatorial formulas for the first order invariants along with proofs of a number of conjectures from [1]. The present work is a further modest contribution to this area. Its content is as follows.

The first three sections concern *long curves* in the oriented plane, also introduced by V. Arnold; a long curve is a generic immersion of the oriented line which coincides with the $x$-axis outside some disc. The equivalence relation is induced by compactly supported plane diffeomorhisms and by the oriented self-tangency moves — see Fig. 5.

In the first section we assign signs and indices to the double points of a long curve and show how to compute the rotation number and the $St$ invariant in terms of the double points (the author learned the idea of this construction from O. Viro; it plays an essential role in [9]).

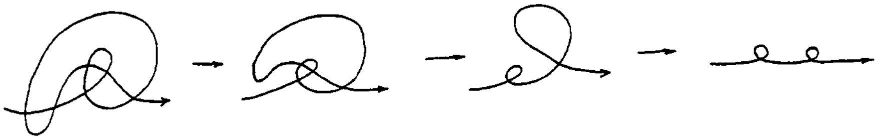


Fig. 5

In the second section we construct an invariant $I$ of long curves with the values in the free group with two generators $F_2$. This invariant is obtained by factorizing the Gauss word of a curve. We show that $I$ is not injective and find its range: it is a normal subgroup of $F_2$ whose quotient group is $\mathbf{Z}$. The origin of the invariant $I$ is a certain vertex model of statistical mechanics. $I$ is determined by an infinite sequence of finite type invariants; this is explained and proved in the forthcoming paper by M. Khovanov [8], in which also a "doodle-valued" generalization of the invariant $I$ is given.

We show in the third section that the invariant $St$ can be reconstructed from $I$ (compare to the paper [9] which was not accessible to the author when the present work was written). Namely, $I$ $(\gamma)$ is interpreted as a path on the unit square grid, and $St$ $(\gamma)$ is twice the signed area, bounded by this path and the main diagonal. Using this relation we prove an Arnold's conjecture concerning the maximal value of $St$ on curves with a fixed number of double points which, in this setting, amounts to the Dido's isoperimetric inequality on the grid. This conjecture, along with other conjectures, is also proved by Shumakovich in [9].

The last section concerns collections of nonoriented immersed closed curves with transversal double points on a nonoriented sphere; the equivalence relation is induced by the sphere diffeomorhisms and the nonoriented self-tangency moves. We call an equivalence class a *rondle* — see Fig. 6 for an example.

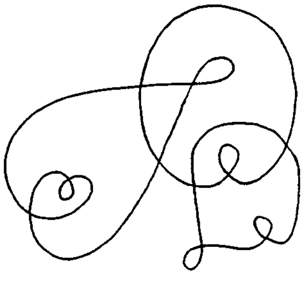


Fig. 6

Rondles are different from smooth doodles in that the move of deleting a curl is allowed for doodles and prohibited for rondles — see Fig. 7.

We construct a one-variable polynomial invariant of rondles by way of a model of statistical mechanics and study its properties. M. Khovanov showed in [8] that

Fig. 7

this invariant can be constructed via a certain representation of the *twin group* introduced in [7].

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## 1. The Invariant *St* for Long Curves

Define two commuting involutions on the set of long curves. Let $\gamma$ be a long curve.

**Definition.** $\bar{\gamma}$ is obtained from $\gamma$ by the reflection in the $x$-axis. $\gamma^*$ is obtained from $\gamma$ by the reflection in the origin with the orientation from left to right.

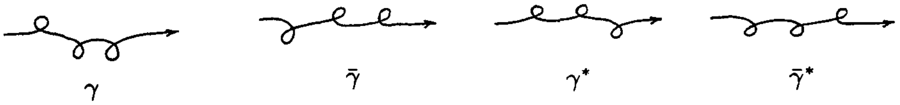


Fig. 8

Consider a generic long curve $\gamma$. Let $A$ be a transversal double point. Traversing $\gamma$ from left to right one visits $A$ twice.

**Definition.** *sign* $A$ is $\pm 1$ according to the orientation of the frame made by tangent vectors to $\gamma$ at the first and the second visits to $A$.

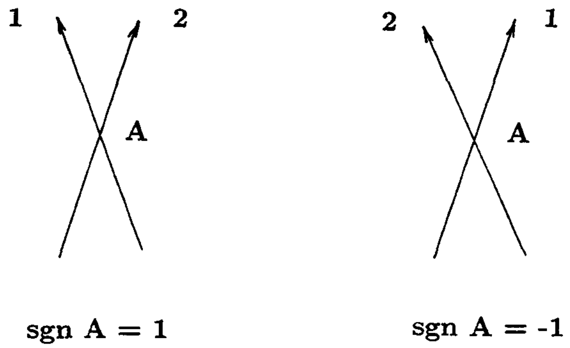


Fig. 9

The rotation $r(\gamma)$ is defined similarly to the case of closed curves: it is the algebraic number of turns made by a tangent vector to $\gamma$ as one traverses it from left to right — see Fig. 10.

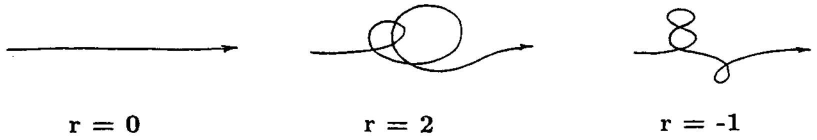


Fig. 10

Notice that $r(\bar{\gamma}) = r(\gamma^*) = -r(\gamma)$.

**Lemma 1.1.** (*Whitney* [13])

$$r(\gamma) = \sum sgn\ A,$$

*summation over the double points $A$ of $\gamma$.*

**Proof.** (see [1]). Observe that the right-hand side does not change under the self-tangency moves: the two double points involved have opposite signs. Likewise, this sum is invariant under the triple point move. Applying these moves one transforms $\gamma$ to the canonical form shown in Fig. 11. All double points are positive for the first curve and negative for the second, hence the formula holds for these curves, and therefore, for $\gamma$. □

Fig. 11

We adjust the definition of the invariant $St$ in our setting as follows.

**Definition.** $St$ is an integer-valued invariant of long curves, which does not change under the oriented self-tangency moves, increases by 2 under a positive triple point move and normalized to have zero value on the $x$-axis and the values $p$ and $q$ on the cannonical curves from Fig. 11.

Let $A$ be a point not on a long curve $\gamma$.

**Definition.** The index *ind* $A$ is the algebraic number of half-turnes made by the vector from $A$ to a point, traversing $\gamma$ from left to right.

The index is locally constant in the complement of $\gamma$; it equals $-1$ in the unbounded component below $\gamma$, and it increases by 2 under a positive crossing of $\gamma$ shown in Fig. 12.

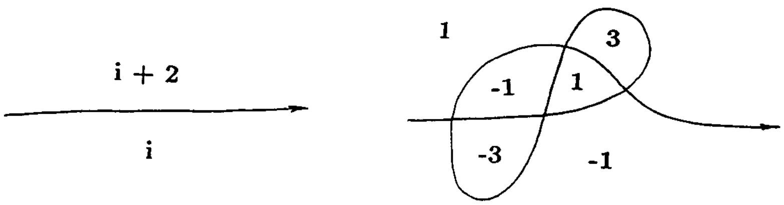
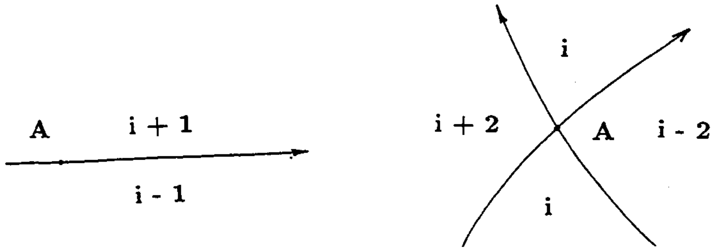


Fig. 12

**Definition.** The index of a smooth point $A$ of a long curve $\gamma$ is the arithmetic mean of the indices assigned to points on both sides of $\gamma$ near $A$. The index of a double point $A$ is the arithmetic mean of the four indices assigned to the four components of the complement of $\gamma$ near $A$ — see Fig. 13.



ind A = i    ind A = i

Fig. 13

A version of the next observation is made in [9]; it was communicated to the author by O. Viro.

**Lemma 1.2.**

$$St\ (\gamma) = \sum ind\ A \cdot sgn\ A,$$

*summation over the double points $A$ of $\gamma$.*

**Proof.** Observe first that the sum on the right-hand side does not change under the oriented self-tangency moves: the two double points have equal indices and opposite signs. Secondly, a direct computation shows that this sum increases by 2 under a positive triple point move (see Fig. 14 for one of the 8 cases to be considered; see Fig. 17 of [1] for the remaining cases). Finally the sum equals $St$ on the cannonical curves from Fig. 11. □

It follows that $St$ assumes the same values on the curves $\gamma, \bar{\gamma}$ and $\gamma^*$.

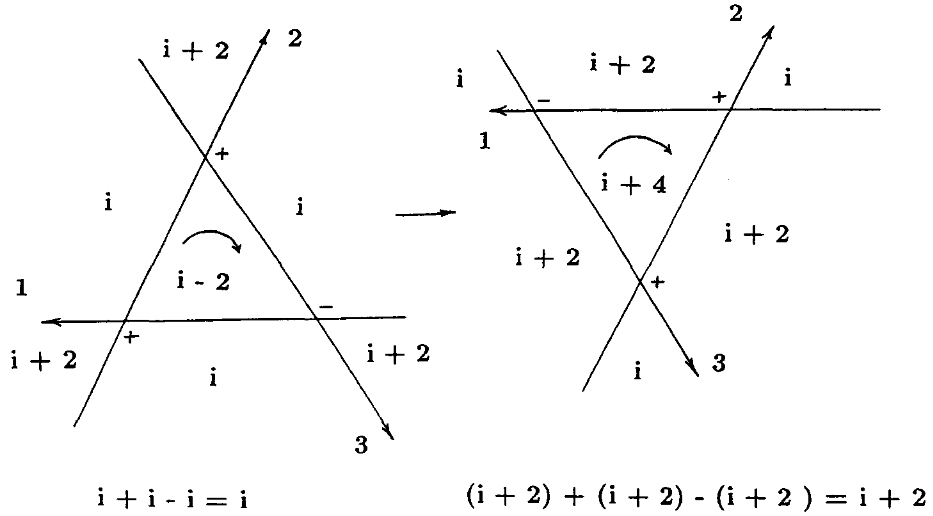


Fig. 14

**Remarks.**

1. The formula for $St$ on closed curves discovered in [9] reads:

$$St\ (\gamma) = \sum ind\ A \cdot sgn\ A + (ind\ B)^2 - 1/4.$$

Here $B$ is a point of the curve; its index is the arithmetic mean of the indices of points on both sides of $\gamma$ near $B$; the signs are assigned traversing the curve from $B$. The index of a point not on a closed curve is the number of full turns of the vector from this point to a point making a complete circuit of the curve. Shumakovich also found two similar formulas that give $St$ in terms of the indices and signs of arcs of the curve and connected components of its complement.

2. One can generalize the invariant $St$ on the long curves:

$$St_k\ (\gamma) = \sum (ind\ A)^k \cdot sgn\ A.$$

This is also a first order invariant.

3. For closed curves $St\ (\gamma)$ can be interpreted as the self-linking number of $\gamma$. Define $\gamma \cap \gamma$ to be the 0-chain in which a double point $A$ has the coefficient $sgn\ A$. Remove this chain from $\gamma$ by a small translation of each double point $A$ into the angle spanned by tangent vectors to $\gamma$ at $A$ — see Fig. 15. Then

$$St\ (\gamma) = Lk\ (\gamma \cap \gamma, \gamma),$$

where the linking number of a point $A$ and a curve $\gamma$ is the algebraic number of the intersection points of $\gamma$ with a generic path from $A$ to infinity. A slight modification of this construction applies to long curves as well.

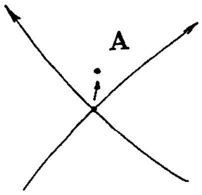


Fig. 15

## 2. An $F_2$-Valued Invariant for Long Curves

We attempt to construct more invariants of the long curves by way of a vertex model of statistical mechanics. Consider a long curve $\gamma$ as an oriented graph, whose vertices are the double points of $\gamma$ and whose edges are the arcs of $\gamma$. Color each edge by a color from $\{1, ..., n\}$. Assign a numerical weight to each double point according to its sign and the colors of the edges, incident to it as shown in Fig. 16.

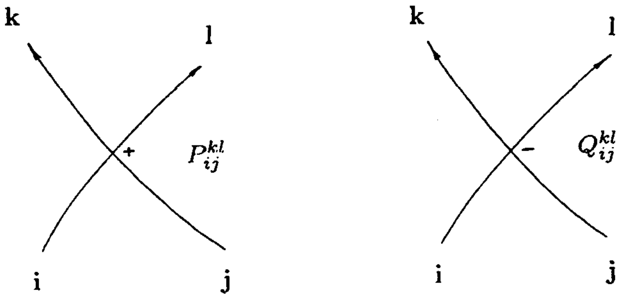


Fig. 16

The state sum $Z\ (\gamma)$ is the sum over all edge colorings of the products of the corresponding weights assigned to the vertices. The following lemma is proved by a direct computation.

**Lemma 2.1.** *$Z\ (\gamma)$ is invariant under the oriented self-tangency moves if and only if the weights satisfy the equalities:*

$$\sum P_{ij}^{uv} Q_{uv}^{kl} = \delta_{ik}\delta_{jl}, \quad \sum P_{ui}^{vl} Q_{vj}^{uk} = \delta_{ik}\delta_{jl}$$

*for all $i, j, k, l$.*

Here, as usual, the summation goes over the repeated indices; $\delta$ is the Kronecker symbol.

Consider an $n$-dimensional vector space $V$ with the basis $\{1, \ldots, n\}$. The tensors $P$ and $Q$ are interpreted as linear transformations $V \otimes V \to V \otimes V$:

$$P(i \otimes j) = \sum P_{ij}^{kl}\ (k \otimes l), \quad Q(i \otimes j) = \sum Q_{ij}^{kl}\ (k \otimes l).$$

We failed to solve the equations on $P$ and $Q$ from Lemma 2.1; instead we make the following observation.

**Lemma 2.2.** *The group* $GL(V) \times GL(V) \times GL(V) \times GL(V)$ *acts on the solutions to these equations via*

$$P \to (A \otimes B)\, P\, (C^{-1} \otimes D^{-1}), \quad Q \to (C \otimes D)\, Q\, (A^{-1} \otimes B^{-1});$$

*here* $A, B, C, D \in GL(V)$.

The proof is a direct computation, and we omit it.

Notice that the above system of equations on $P$ and $Q$ has a simple solution $P = Q = \sigma$ where

$$\sigma(x \otimes y) = y \otimes x$$

(no interaction at a vertex). Acting on this solution by $GL(V)^4$ one obtains new solutions:

$$P = (X \otimes Y)\, \sigma, \quad Q = (Y^{-1} \otimes X^{-1})\, \sigma,$$

where $X = AD^{-1}, Y = BC^{-1}$. Thus

$$P_{ij}^{kl} = X_j^k\, Y_i^l, \; Q_{ij}^{kl} = (Y^{-1})_j^k\, (X^{-1})_i^l,$$

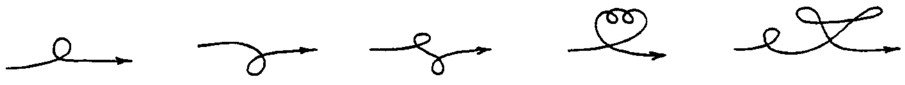
where $X, Y : V \to V$ are arbitrary nondegenerate linear operators.

Fix the colors $i$ and $j$ of the left and right ends of $\gamma$, and let $Z_{ij}(\gamma)$ be the state sum over all colorings of the remaining edges. Thus $Z\,(\gamma)$ is interpreted as a linear operator $V \to V$. The operator $Z\,(\gamma)$ is a product of the operators $X, X^{-1}, Y, Y^{-1}$; there are $n$ symbols $X$ and $X^{-1}$, and $n$ symbols $Y$ and $Y^{-1}$ in this product, where $n$ is the number of double points of $\gamma$. The product is obtained according to the following rule:

> Traverse $\gamma$ from left to right. Each time one passes a double point, multiply the previously constructed product by a new term on the right. This term is $X^{sgn\ A}$ at the first, and $Y^{sgn\ A}$ at the second visit of a double point $A$.

Replace the operators $X$ and $Y$ in this product by the abstract symbols $x$ and $y$ which generate the free group $F_2$.

**Definition.** The invariant $I\,(\gamma)$ is an element of $F_2$ constructed according to the above rule. See Fig. 17 for some examples.



$xy$ $\quad$ $x^{-1}y^{-1}$ $\quad$ $xyx^{-1}y^{-1}$ $\quad$ $x^2yxy^2$ $\quad$ 1

Fig. 17

A few things follow immediately from the definition. First, $I\ (\gamma)$ is invariant under the oriented self-tangency moves. Secondly, $I\ (\gamma)$ is multiplicative under the concatenation (connected summation) of the long curves. Thirdly, the invariant $I\ (\gamma)$ is not injective, as the last curve in Fig. 17 shows (that this curve is nontrivial could be shown the same way one proves that the connected sum of nontrivial knots is knotted — see p. 57 of [6]).

Note the following relations between $I\ (\gamma), I(\bar{\gamma})$ and $I(\gamma^*)$. To obtain $I(\bar{\gamma})$ from $I\ (\gamma)$ one makes the substitution $x \leftrightarrow x^{-1}, y \leftrightarrow y^{-1}$; to obtain $I(\gamma^*)$ from $I\ (\gamma)$ one reads $I\ (\gamma)$ backwards, making the substitution $x \leftrightarrow y^{-1}, x^{-1} \leftrightarrow y$.

Note also that, according to Lemma 1.1, $r(\gamma)$ is the image of $I\ (\gamma)$ in $\mathbf{Z}$ under the homomorphism

$$\phi : F_2 \to \mathbf{Z},\ \ \phi(x) = t,\ \ \phi(y) = 1$$

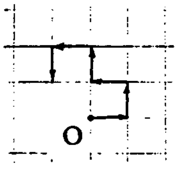
(or the homomorphism

$$\psi : F_2 \to \mathbf{Z},\ \ \psi(x) = 1,\ \ \psi(y) = t).$$

**Remark.** (M. Khovanov [8]). The invariant $I\ (\gamma)$ can be generalized to an invariant that takes values in $\pi_1(D(\gamma)) * F_2$ where $D(\gamma)$ is the long curve $\gamma$ considered as a doodle, and $\pi_1$ is the fundamental group of a doodle, defined by Khovanov in [7].

We regard $I\ (\gamma)$ as a path on the unit square grid which is the Cayley graph of the group $F_2$.

**Definition.** $P\ (\gamma)$ is a path on the unit square grid, starting at the origin and determined by $I\ (\gamma)$ as follows: read $I\ (\gamma)$ from left to right and interpret $x, x^{-1}, y$ and $y^{-1}$ as one step left, right, up or down, respectively — see Fig. 18.



$xyx^{-1}yx^{-1}y^{-1}$

Fig. 18

Note that $P(\bar{\gamma})$ is centrally symmetric to $P\ (\gamma)$ with respect to the origin, while $P(\gamma^*)$ is obtained from $P\ (\gamma)$ by traversing it backwards and reflecting in the main north-east diagonal through the origin.

Consider the homomorhpism

$$f : F_2 \to \mathbf{Z},\ \ f(x) = t,\ \ f(y) = t^{-1};$$

let $G \subset F_2$ be its kernel. Since a positive double point contributes one $x$ and one $y$, while a negative one — one $x^{-1}$ and one $y^{-1}$ to $I\ (\gamma)$, we see that $I\ (\gamma) \in G$.

Accordingly, the terminal point of the path $P\ (\gamma)$ lies on the main diagonal. The signed distance from the terminal point to the origin, measured in the diagonals of unit squares, equals the rotation number $r(\gamma)$.

**Lemma 2.3.** *Let $P$ be a path on the grid from the origin to a point on the main diagonal, in which no unit segment is traversed consecutively back and forth. Then there exists a long curve $\gamma$ such that $P = P\ (\gamma)$.*

**Proof.** Induction in the length of $P$. If $P$ intersects the main diagonal then it is a composition of two shorter paths $P_1$ and $P_2$ with the end points on the diagonal. By the induction assumption, $P_1 = P(\gamma_1), P_2 = P(\gamma_2)$ for some long curves $\gamma_1$ and $\gamma_2$. Then $P = P(\gamma_1 \sharp \gamma_2)$ where $\sharp$ denotes the connected sum.

Assume that $P$ lies below the main diagonal (the other case is dealt with similarly). Consider the diagonal, obtained from the main one by the translation one unit to the right; let $Q$ be the path which cuts from $P$ — see Fig. 19. Then, by the induction assumption, $Q = P\ (\gamma)$ for some long curve $\gamma$.

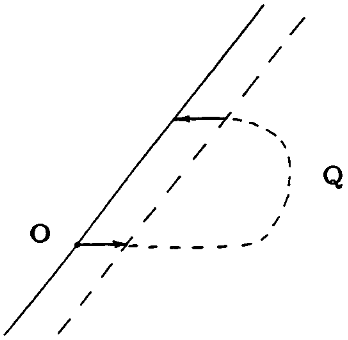


Fig. 19

There are 4 cases to consider:

$$P = x\ Q\ x^{-1},\ P = y^{-1}\ Q\ y,\ P = x\ Q\ y,\ P = y^{-1}\ Q\ x^{-1}.$$

In each case $P = P\ (\Gamma)$ where the long curve $\Gamma$ is obtained from $\gamma$ as shown in Fig. 20.

We obtain the following corollary.

**Theorem 2.4.** *The range of the invariant $I\ (\gamma)$ is the subgroup $G \subset F_2$.*

## 3. Reconstructing $St(\gamma)$ from $I(\gamma)$

Given a long curve $\gamma$ let $\tilde{I}\ (\gamma)$ be its encoding word in $x, x^{-1}, y, y^{-1}$ (before cancelations), and $\tilde{P}\ (\gamma)$ — the corresponding path on the grid (which may be backtracking).

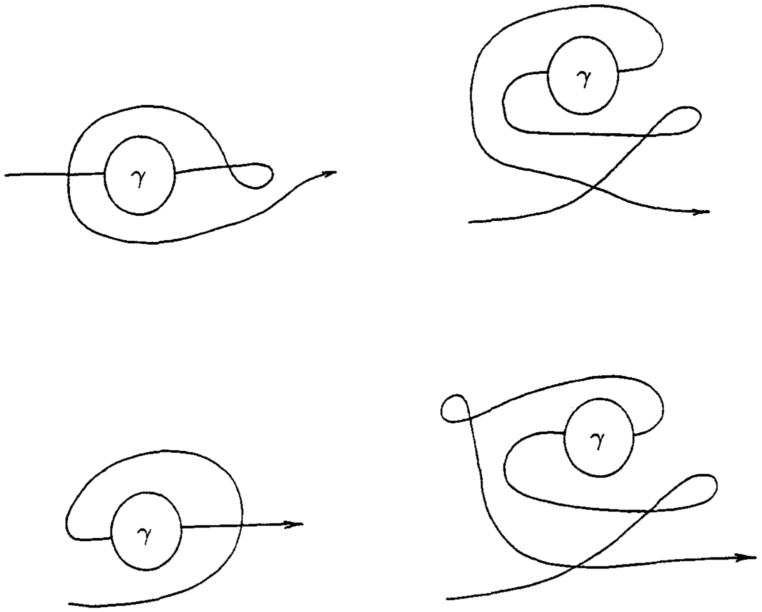


Fig. 20

Every unit oriented segment in $\tilde{P}\ (\gamma)$ corresponds to a double point of $\gamma$; this correspondence is 2-to-1.

A positive double point is encoded by $x$ and $y$, a negative one by $x^{-1}$ and $y^{-1}$. Thus the sign of a double point is the sign of the projection of the corresponding unit segment of $\tilde{P}\ (\gamma)$ onto the north-east direction — see Fig. 21.

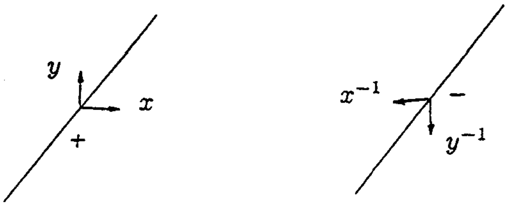


Fig. 21

Likewise, one can recover the index of a double point. Traverse $\gamma$ from left to right. The indices of points on $\gamma$ sufficiently far to the left are equal to zero. The index of a current point of $\gamma$ changes under crossing a double point as shown in Fig. 22.

Thus each symbol $x$ and $y^{-1}$ in $\tilde{I}\ (\gamma)$ increases the index of the current point of $\gamma$ by 2, while $x^{-1}$ and $y$ decrease it by 2. The index of a double point is the mean value of the indices of the points immediately before and after it along $\gamma$. Therefore the index of a double point, corresponding to a unit segment in $\tilde{P}\ (\gamma)$, is

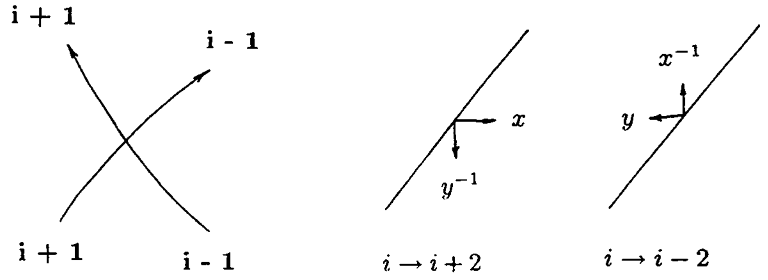
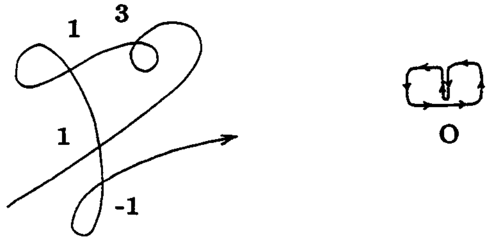


Fig. 22

the signed distance from the center of this segment to the main diagonal, positive below and negative above it, measured in quaters of the diagonal of a unit square — see Fig. 23. It follows that the index of a double point is always odd.



$$\tilde{I} = x\ x\ y\ x^{-1}\ y^{-1}\ y\ \ x^{-1}\ y^{-1}$$
$$i = 1, 3, 3, 1, \quad 1, \quad 1, -1, -1$$

Fig. 23

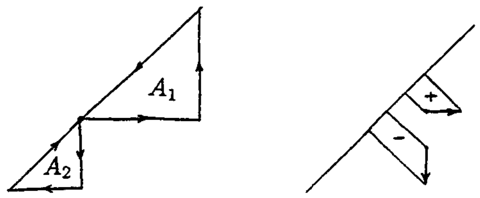
Connect the terminal point of $\tilde{P}$ $(\gamma)$ to the origin by a segment of the main diagonal. Let $A$ $(\gamma)$ be the signed area enclosed by thus obtained closed path — see Fig. 24. Notice that $A$ $(\gamma)$ depends on $P$ $(\gamma)$ only.

**Theorem 3.1.**

$$St\ (\gamma) = 2A\ (\gamma).$$

**Proof.** Consider a unit segment of $\tilde{P}$ $(\gamma)$, and let $B$ be the corresponding double point of $\gamma$. The area of the trapezoid, shown in Fig. 24(b), equals $d/4$, where $d$ is the distance from the center of the segment to the main diagonal, measured in quaters of the diagonal of a unit square. According to the above discussion, $d = ind\ B$.

The sign of the area of the trapezoid depends on whether the unit segment lies above or below the main diagonal, and on the direction of its projection on this



$A_1 = \frac{9}{2} \quad A_2 = -2$

a) b)

Fig. 24

diagonal. The former is determined by the sign of $d$, and the latter, according to the previous discussion, by $sgn\ B$. Hence the signed area of the trapezoid is $sgn\ B \cdot ind\ B/\ 4$.

Since each double point contributes two segments to $\tilde{P}\ (\gamma)$, summation over the double points of $\gamma$ yields:

$$A\ (\gamma) = \frac{1}{2} \sum sgn\ B \cdot\ ind\ B.$$

According to Lemma 1.2, the right-hand side is $St\ (\gamma)/2$. □

Let $\gamma$ be a long curve with $n$ double points. What is the maximal value of $St(\gamma)$ on such curves? The answer, conjectured by V. Arnold in [1], is adjusted in our setting as follows.

**Theorem 3.2.** *$St\ (\gamma) \leq n^2$, with equality only for the curves $\gamma$ shown in Fig. 25 and the symmetric curves $\bar{\gamma}$.*

Fig. 25

**Proof.** The path $\tilde{P}\ (\gamma)$ has the length $2n$. We are interested in the grid version of the Dido's isoperimetric problem: what is the maximal signed area $A\ (\gamma)$ bounded by $\tilde{P}\ (\gamma)$ and the main diagonal? We claim that $A\ (\gamma) \leq n^2/2$ with equality only for the paths shown in Fig. 26.

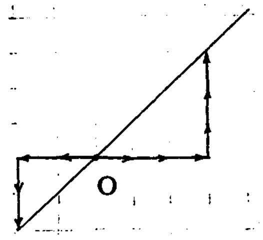


Fig. 26

The proof is by induction in $n$. If $\tilde{P}\ (\gamma)$ intersects the main diagonal at points other than its terminal ones, it can be partitioned into shorter paths of lengths $n_1$ and $n_2$, for which the inequality holds by the induction assumption. Since $n_1^2 + n_2^2 < (n_1 + n_2)^2$, we are done in this case.

Without loss of generality assume that $\tilde{P}\ (\gamma)$ lies below the diagonal. The following moves do not decrease the area enclosed by $\tilde{P}\ (\gamma)$ and do not increase its length — see Fig. 27.

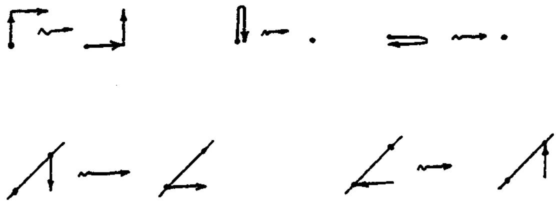


Fig. 27

Successively applying these moves one changes $\tilde{P}\ (\gamma)$ to the path shown in Fig. 26, which bounds the area $n^2/2$. Hence $St\ (\gamma) = 2A\ (\gamma) \leq n^2$.

If $St\ (\gamma) = n^2/2$ then $\tilde{P}\ (\gamma) = x^n y^n$. Therefore all double points are positive, and, by Lemma 1.1, $r(\gamma) = n$, the maximal rotation number for long curves with $n$ double points. Such extremal curves were classified by Arnold in [1]: the greatest value of $St$ is attained on the curves from Fig. 25. $\square$

Let $\gamma$ be a long curve that has $p$ positive and $q$ negative double points. How great can $St(\gamma)$ be? The following inequality is an adjustment of a result by Shumakovich ([9]).

**Theorem 3.3.**

$$St(\gamma) \leq p^2 + q^2$$

*(see Fig. 28 for an example of equality).*

**Proof.** We repeat the argument from [9] in our setting.

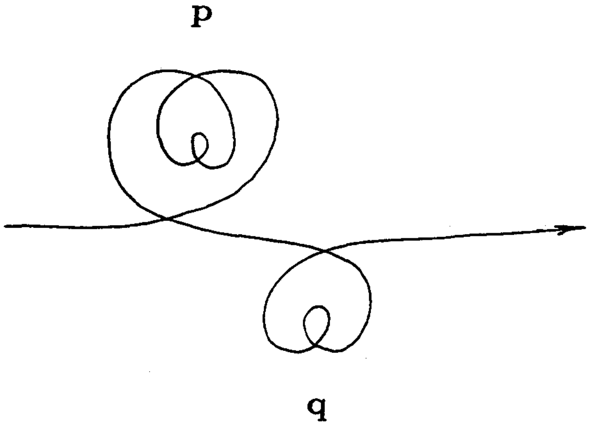


Fig. 28

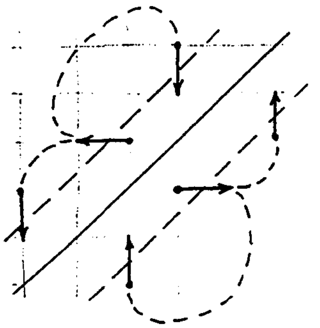


Fig. 29

Consider the path $\tilde{P}(\gamma)$. It starts and ends on the main diagonal. A double point of $\gamma$ contributes two unit segments to $\tilde{P}(\gamma)$, one horizontal and one vertical; the former precedes the latter in $\tilde{P}(\gamma)$. If the path intersects some diagonal parallel to the main one then its first intersection with this diagonal is in the direction from the main diagonal — see Fig. 29. Therefore if there exists a double point of $\gamma$ with some positive index then there exists a positive double point with the same index; likewise, if there is a double point with some negative index then there is a negative double point with this index.

Let $a_i$, $i \geq 1$ be the number of positive double points with the index $2i-1$, and $b_i$, $i \geq 1$, the number of negative double points with the index $-(2i-1)$. Then $\sum a_i \leq p$ and $\sum b_j \leq q$. Let $2M-1$ be the maximal and $-(2m-1)$ the minimal index of double points of $\gamma$. Then $\tilde{P}(\gamma)$ is contained in the strip bounded by the diagonals, parallel to the main one, at distance $2M$ and $2m$ from it. Hence $\tilde{P}(\gamma)$ intersects all diagonals contained in this strip. According to the preceding paragraph, $a_i \geq 1$ for $i = 1, \ldots, M$ and $b_j \geq 1$ for $j = 1, \ldots, m$.

We conclude by a chain of inequalities:

$$\begin{aligned}
St(\gamma) &\le \sum_{sgnB\cdot indB>0} sgn\ B \cdot ind\ B \\
&= \sum_{i=1}^{M}(2i-1)\ a_i + \sum_{j=1}^{m}(2j-1)\ b_j \\
&= \sum_{i=1}^{M}(2i-1) + \sum_{i=1}^{M}(2i-1)\ (a_i-1) + \sum_{j=1}^{m}(2j-1) + \sum_{j=1}^{m}(2j-1)\ (b_j-1) \\
&\le M^2+m^2+2M\ \sum_{i=1}^{M}(a_i-1) + 2m\ \sum_{j=1}^{m}(b_j-1) \\
&\le (M+\sum_{i=1}^{M}(a_i-1))^2 + (m+\sum_{j=1}^{m}(b_j-1))^2 \\
&= (\sum_{i=1}^{M}a_i)^2 + (\sum_{j=1}^{m}b_j)^2 \le p^2+q^2
\end{aligned}$$

(with an equality only if each $a_i$ and $b_j$ is equal to 1). □

It would be interesting to study inequalities relating the generalized invariants $St_k$.

## 4. A Polynomial Invariant for Rondles

Given a collection of oriented plane or spherical curves, considered modulo the oriented self-tangency moves, one attempts to construct their invariants along the lines at the beginning of Sec. 2. Since in this case one cannot assign signs to double points, one sets: $P = Q$. The tensor $P$ satisfies the equations:

$$\sum P_{ij}^{uv} P_{uv}^{kl} = \delta_{ik}\delta_{jl}, \quad \sum P_{ui}^{vl} P_{vj}^{uk} = \delta_{ik}\delta_{jl}$$

This system can be solved manually in the case of two colors, but its solutions do not produce any nontrivial invariants.

Therefore we change the model: we consider collections of nonoriented spherical curves with transversal double points modulo the nonoriented self-tangency moves, that is, rondles. Color each connected component of the complement of the curves by a color from $\{1, \dots, n\}$. Assign a weight to each vertex (i.e. double point) as shown in Fig. 30; the tensor $W$ is symmetric with respect to the cyclic permutation of its indices: $W_{ij}^{kl} = W_{jl}^{ik} = W_{lk}^{ji} = W_{ki}^{lj}$.

The state sum is

$$Z = \sum_{colorings} \prod_{vertices} W_{ij}^{kl}.$$

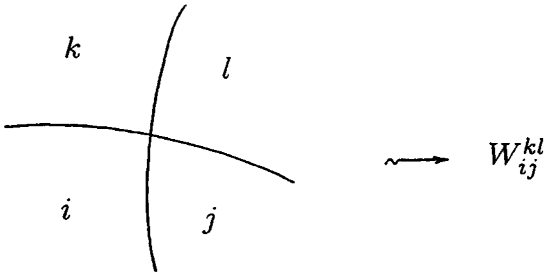


Fig. 30

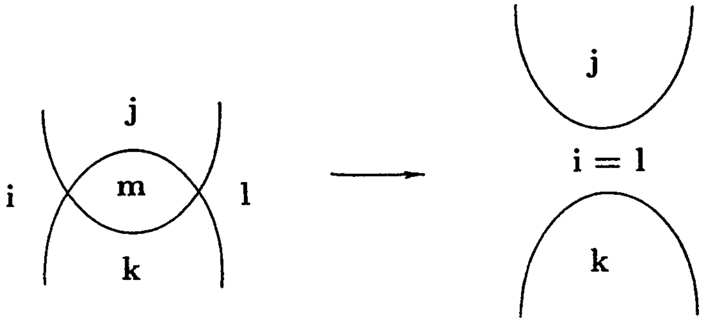
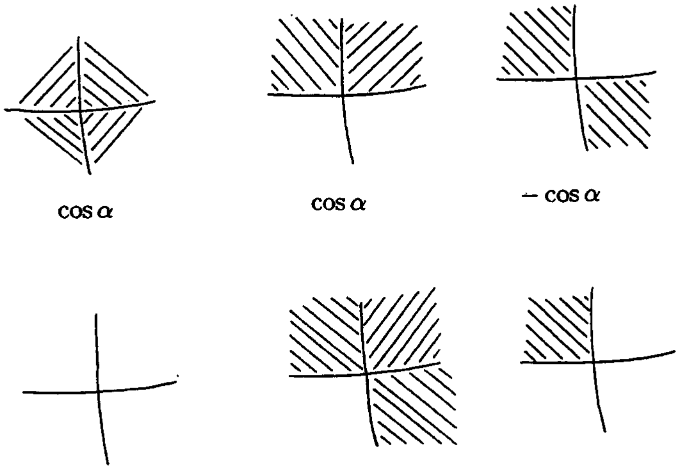


Fig. 31



$\cos\alpha$ $\cos\alpha$ $-\cos\alpha$

$\cos\alpha$ $\sin\alpha$ $-\sin\alpha$

Fig. 32

We want it to be invariant under the self-tangency move. The equations to satisfy are

$$\sum_m W_{ik}^{jm} W_{kl}^{mj} = \delta_{il} \quad \text{for all } i, j, k, l$$

(analogue of Lemma 3.1) — see Fig. 31.

We did not attempt to solve these equations in general. We present here a solution for two colors which leads to a nontrivial one variable polynomial for rondles.

The colors used are black and white. The weights are specified in Fig. 32; $\alpha$ is a parameter.

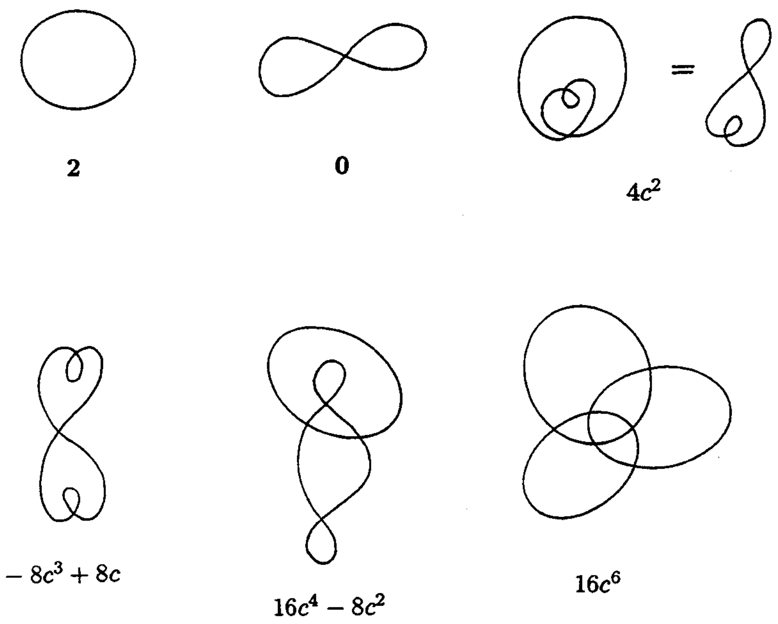
Denote the state sum by $Z(\alpha)$.

**Theorem 4.1.** *$Z(\alpha)$ is invariant under the nonoriented self-tangency move.*

The proof is a direct computation, and we omit it. Note that the same weights produce an invariant of rondles on any surface, not necessarily oriented.

A simultaneous change of colors to the opposite amounts to changing $\alpha$ to $-\alpha$. Hence $Z(\alpha)$ is an even function of $\alpha$, and therefore a polynomial in $\cos\alpha$. Denote this polynomial by $J\ (\gamma)$ where $\gamma$ is a rondle. We shorthand $\cos\alpha$ to $c$.

A few examples are shown in Fig. 33.



$2$

$0$

$4c^2$

$-8c^3 + 8c$

$16c^4 - 8c^2$

$16c^6$

Fig. 33

Here is another example. Let $\gamma_{pq}$ be the curve shown in Fig. 34.

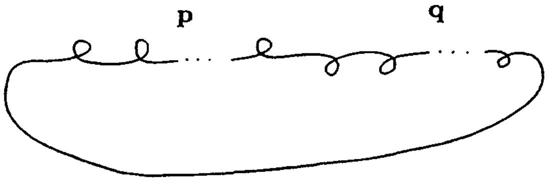


Fig. 34

**Lemma 4.2.**

$$J(\gamma_{pq}) = ((\cos\alpha + \sin\alpha)^p + (-1)^{p+q}(\cos\alpha - \sin\alpha)^p)\ ((\cos\alpha + \sin\alpha)^q + (-1)^{p+q}(\cos\alpha - \sin\alpha)^q)\,.$$

*In particular,* $J\ (\gamma_{pq}) = 0$ *evaluated at* $\alpha = 0$ *if* $p + q$ *is odd.*

The proof is again a direct computation. Notice that the formula is symmetric in $p$ and $q$ — the reason being that $\gamma_{pq}$ is equivalent to $\gamma_{qp}$ on the sphere.

An immersed curve on the sphere does not have a well-defined rotation number; the only invariant of the regular homotopy is the parity of the number of double points (the spherical version of the Whitney theorem). Thus every spherical curve is regular homotopic to either a circle or an eight-shaped curve.

**Lemma 4.3.** *The value of* $J\ (\gamma)$ *at* $\alpha = 0$ *is invariant under the triple point move. This value is zero if* $\gamma$ *has at least one component with an odd number of self-intersection points; otherwise it equals* $2^{n+1}$ *where* $n$ *is the number of components of the rondle* $\gamma$ *— compare with Fig. 33.*

**Proof.** Applying the self-tangency and the triple point moves one changes $\gamma$ to a disjoint union of several eight-shaped curves and circles — see Fig. 35. If at least one eight-shaped curve is present, $J\ (\gamma) = 0$; otherwise $J\ (\gamma) = 2^{n+1}$ where $n$ is the number of circles. □

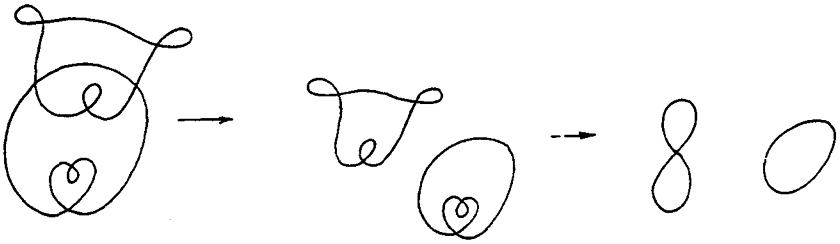


Fig. 35

**Remark.** M. Khovanov points out in [8] that a rondle is the closure of a *twin*; twins, introduced in [7], are analogues of braids. Khovanov constructs the invariant $J(\gamma)$ via a certain representation of the twin group.

We conclude with a few problems:

1. Find reasonable skein relations for $J(\gamma)$.
2. Investigate the case of more than two colors.
3. Are there relations between $J(\gamma)$ and $I(\gamma)$ for a one-component rondle?

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