

ON ALGEBRAICALLY INTEGRABLE OUTER BILLIARDS

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We prove that if the outer billiard map around a plane oval is algebraically integrable in a certain nondegenerate sense then the oval is an ellipse.

In this note, an outer billiard table is a compact convex domain in the plane bounded by an oval (closed smooth strictly convex curve) C . Pick a point x outside of C . There are two tangent lines from x to C ; choose one of them, say, the right one from the viewpoint of x , and reflect x in the tangency point. One obtains a new point, y , and the transformation $T : x \mapsto y$ is the outer (also called dual) billiard map. We refer to [Tabachnikov and Dogru 2005; Tabachnikov 1995; 2005] for surveys of outer billiards.

If C is an ellipse then the map T possesses a 1-parameter family of invariant curves, the homothetic ellipses; these invariant curves foliate the exterior of C . Conjecturally, if an outer neighborhood of an oval C is foliated by the invariant curves of the outer billiard map, then C is an ellipse — this is an outer version of the famous Birkhoff conjecture concerning the conventional, inner billiards.

In this note we show that ellipses are rigid in a much more restrictive sense of algebraically integrable outer billiards; see [Bolotin 1990] for the case of inner billiards.

We make the following assumptions. Let $f(x, y)$ be a (nonhomogeneous) real polynomial that has zero as a nonsingular value and such that C is a component of its zero level curve. Thus f is the defining polynomial of the curve C , and if a polynomial vanishes on C , then it is a multiple of f (see for example [Clemens 2003; Fischer 2001; Walker 1978]). Assume that a neighborhood of C is foliated by invariant curves of the outer billiard map T and that this foliation is algebraic in that its leaves are components of the level curves of a real polynomial $F(x, y)$. Since C itself is an invariant curve, we assume that $F(x, y) = 0$ on C and that dF is not identically zero on C . Thus $F(x, y) = g(x, y)f(x, y)$ where $g(x, y)$ is a polynomial not identically zero on C . Under these assumptions, our result is as follows.

Theorem 1. *C is an ellipse.*

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Proof. Consider the tangent vector field $v = F_y \partial/\partial x - F_x \partial/\partial y$ (the symplectic gradient) along C . This vector field is nonzero (except at possibly a finite number of points) and tangent to C . The tangent line to C at point (x, y) is given by $(x + \varepsilon F_y, y - \varepsilon F_x)$, and the condition that F is T -invariant means that the function

$$(1) \quad F(x + \varepsilon F_y, y - \varepsilon F_x)$$

is even in ε for all $(x, y) \in C$. Expand in a series in ε ; the first order term in ε vanishes automatically and the first nontrivial condition is cubic in ε :

$$(2) \quad W(F) := F_{xxx} F_y^3 - 3F_{xxy} F_y^2 F_x + 3F_{xyy} F_y F_x^2 - F_{yyy} F_x^3 = 0$$

on C . We claim that this already implies that C is an ellipse. The idea is that otherwise the complex curve $f = 0$ would have an inflection point, in contradiction with identity (2).

Consider the polynomial

$$H(F) = \det \begin{pmatrix} F_y & -F_x \\ F_{yy} F_x - F_{xy} F_y & F_{xx} F_y - F_{xy} F_x \end{pmatrix}.$$

Lemma 2. (i) $v(H(F)) = W(F)$.

(ii) $H(F) = H(gf) = g^3 H(f)$ on C .

(iii) If C' is a nonsingular algebraic curve with a defining polynomial $g(x, y)$, then $H(g)(x, y) = 0$ if and only if (x, y) is an inflection point of C' .

Proof. The first two claims follow from straightforward computations. To prove the third, note that $H(g)$ is the second order term in ε of the Taylor expansion of the function $g(x + \varepsilon g_y, y - \varepsilon g_x)$; see (1). Hence $H(g) = 0$ at the points where the tangent line is second order tangent to the curve, that is, at the inflection points. \square

It follows from Lemma 2 and (2) that $H(F) = \text{const}$ on C . Indeed, $v(H(F)) = W(F) = 0$; hence the directional derivative of $H(F)$ along C is zero. Since C is convex, $H(F) \neq 0$. Indeed, if $H(F) = 0$ then, by Lemma 2, $H(f) = 0$ and all points of C are its inflections. Thus we may assume that $H(F) = 1$ on C . It follows that $g^3 H(f) - 1$ vanishes on C and hence

$$(3) \quad g^3 H(f) - 1 = hf,$$

where $h(x, y)$ is some polynomial.

Now consider the situation in $\mathbb{C}\mathbb{P}^2$. We use the notation C' for the complex algebraic curve given by the homogenized polynomial $\tilde{f}(x : y : z) = f(x/z, y/z)$. Unless C is a conic, this curve has inflection points (not necessarily real). Let d be the degree of C' .

Lemma 3. *Not all the inflections of C' lie on the line at infinity.*

Proof. Consider the Hessian curve given by

$$\det \begin{pmatrix} \bar{f}_{xx} & \bar{f}_{xy} & \bar{f}_{xz} \\ \bar{f}_{yx} & \bar{f}_{yy} & \bar{f}_{yz} \\ \bar{f}_{zx} & \bar{f}_{zy} & \bar{f}_{zz} \end{pmatrix} = 0.$$

The intersection points of the curve C' with its Hessian curve are the inflection points of C' (recall that C' is nonsingular). The degree of the Hessian curve is $3(d-2)$ because taking two derivatives lowers the degree by 2 and taking the determinant multiplies terms in threes. By the Bezout theorem, the total number of inflections, counted with multiplicities, is $3d(d-2)$. Also, the order of intersection equals the order of the respective inflection and does not exceed $d-2$, see for example [Walker 1978]. The number of intersection points of C' with a line equals d . Hence the inflection points of C' that lie on a fixed line contribute, at most, $d(d-2)$ to the total of $3d(d-2)$. The remaining inflection points lie off this line. \square

To conclude the proof of [Theorem 1](#), consider a finite inflection point of C' . According to [Lemma 2](#), at such a point we have $f = H(f) = 0$, which contradicts [\(3\)](#). This proves that C is a conic. \square

Remarks. First, it would be interesting to remove the nondegeneracy assumptions in [Theorem 1](#).

Second, a more general version of Birkhoff's integrability conjecture is as follows. Let C be a plane oval whose outer neighborhood is foliated by closed curves. For a tangent line ℓ to C , the intersections with the leaves of the foliation define a local involution σ on ℓ . Assume that, for every tangent line, the involution σ is projective. Conjecturally, then C is an ellipse and the foliation consists of ellipses that form a pencil (that is, share four — real or complex — common points). For a pencil of conics, the respective involutions are projective: this is a Desargues theorem; see [Berger 1987]. It would be interesting to establish an algebraic version of this conjecture.

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