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EXACT TRANSVERSE LINE FIELDS AND PROJECTIVE BILLIARDS IN A BALL

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1 Introduction

Projective billiards generalize the usual ones. The (usual) billiard transformation acts on the set of oriented lines in $\mathbf{R}^n$ intersecting the billiard table bounded by the billiard hypersurface $M^{n-1} \subset \mathbf{R}^n$. This transformation T is defined by the familiar law of geometric optics: the incoming ray r, the outgoing ray $T(r)$ and the normal to M at the impact point lie in one 2-plane, and the angles made by r and $T(r)$ with the normal are equal (see, e.g. [T1] for a survey of mathematical billiards).

To define the projective billiard transformation one needs an extra structure: a smooth field of transverse directions along the hypersurface M. Given such a field the law of the projective billiard reflection reads:

1. *The incoming ray, the outgoing ray and the transverse line at the impact point x lie in one 2-plane π;*
2. *The above three lines and the line of intersection of π with the tangent hyperplane T_xM constitute a harmonic quadruple of lines.*

Recall that four lines through a point is a harmonic quadruple if the cross–ratio of these lines equals -1. The cross–ratio of the lines is the cross–ratio of their intersection points with a fifth line; it does not depend on the choice of this auxiliary line.

Clearly, the projective billiard transformation preserves the lines tangent to M. The projective billiard transformation is equivariant with respect to the projective transformations of $\mathbf{R}^n$; this explains the name "projective billiards". If the transverse field consists of the normals to M then the projective billiard transformation is the usual one. More generally, let g be a metric in a domain, containing M, whose geodesics are straight lines (by a Beltrami theorem g is a metric of constant curvature). Then the billiard in this metric is a projective billiard, the transverse field being the field of g-normals to M.

A study of projective billiards in the plane can be found in [T2]; the present paper extends some of the results to the multidimensional case.

We will be mostly concerned with a very particular case: M is the unit sphere in $\mathbf{R}^n$. Unlike the usual billiard in a ball the projective ones are quite interesting (due to the extra structure, the transverse line field).

The content of the paper is as follows. In section 2 we study a special class of transverse line fields along a hypersurface $M^{n-1} \subset \mathbf{R}^n$, called exact. These fields are in one to one correspondence with exact 1-forms on M. Although defined in Euclidean terms, the class of exact transverse line fields is invariant under the projective transformations of $\mathbf{R}^n$ (the case $n = 2$ was considered in [T3]). We interpret the exactness condition in terms of contact geometry.

In section 3 we construct an invariant symplectic form for the projective billiard transformation in a ball, associated with an exact transverse line field along the boundary sphere. This symplectic form does not depend on a particular choice of an exact transverse field and coincides with the canonical symplectic form on the space of lines in the hyperbolic geometry inside the ball (Klein's model). As a consequence, the usual billiard transformation in an ellipsoid preserves not only the usual symplectic structure on the space of oriented lines, coming from the Euclidean geometry, but also the one associated with the hyperbolic geometry inside the ellipsoid. One may expect the two symplectic structures to be Poisson compatible (the definition is given in section 3) but this does not appear to be the case.

In section 4 we find the generating function for the projective billiard map in a ball, associated with an exact transverse line field, and prove the existence of n-periodic trajectories for every $n \geq 2$.

In the last section we discuss the continuous limit case of the projective billiard transformation, which is a flow in the tangent bundle of a hypersurface M in $\mathbf{R}^n$, equipped with a transverse line field ξ. We call this the projective geodesic flow and its trajectories the projective geodesics. The acceleration vector of a projective geodesic is colinear with ξ at every point. Projective geodesics are the geodesics of a connection on M naturally associated with the transverse field ξ. The relation of the projective geodesic flow to projective billiards is the same as the relation of the geodesic flow to usual billiards.

We use vector notation throughout the paper: if $x, y \in \mathbf{R}^n$ and f is a function on $\mathbf{R}^n$ then

$$xy = x_1y_1 + ... + x_ny_n \ , \quad xdy = x_1dy_1 + ... + x_ndy_n \, ,$$
$$dx \wedge dy = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n \ , \quad f_x = (f_{x_1}, ..., f_{x_n}) \ , \quad \text{etc.}$$

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2 Exact Transverse Line Fields

Let $M^{n-1} \subset \mathbf{R}^n$ be a closed smooth hypersurface and $\nu(x)$, $x \in M$ be the outward unit normal vector field. Every transverse line field ξ along M is generated by a vector field n, normalized so that $n(x)\nu(x) = 1$ identically on M.

Definition. A transverse line field is called exact if the 1-form $n(x)d\nu(x)$ on M is exact, that is, the differential of a function.

Lemma 2.1. *Let the second quadratic form of M be everywhere nondegenerate. Then the correspondence between exact transverse line fields and exact 1-forms on M is one to one.*

Proof. Given an exact 1-form df, one wants to find a transverse vector field n such that $n\, d\nu = df$ and $n\nu = 1$. Let B be the second quadratic form of M; then $B(u,v) = v\, d\nu(u)$ for tangent vectors u and v. Since B is nondegenerate there exists a unique tangent vector field v such that $v\, d\nu = df$. Then $n = \nu + v$ is the desired transverse field.

The next three lemmas provide examples of exact transverse line fields.

Lemma 2.2. *Let $M = \{x \in \mathbf{R}^n \mid x^2 = 1\}$ be the unit sphere, and f be a positive function on $\mathbf{R}^n - 0$, homogeneous of degree $\lambda \neq 0$. Then the line field, generated by the vector field* $\operatorname{grad} f$, *is exact.*

Proof. One has $\nu(x) = x$. Let $n(x) = \operatorname{grad} f(x)/\lambda f(x)$. By Euler's formula, $x f_x = \lambda f$, therefore $n(x)x = 1$. It follows that $n(x)dx = d(\log f^{1/\lambda})$ on M.

Lemma 2.3. *Let M be the unit sphere and f a function on M. Then the field $n(x) = x + \operatorname{grad} f$ is exact. Every exact field is obtained in this way.*

Proof. If $n(x) = x + \operatorname{grad} f$ then $n(x)\, x = 1$, and $n(x)\, dx = df$ on M. Conversely, let $n(x)$ be exact and $n(x)\, x = 1$. Then $v(x) = n(x) - x$ is a tangent vector field on M. Since $x\, dx|_M = 0$, the 1-form $v(x)\, dx$ is exact, that is, $v(x)\, dx = df$ for some function f on M. Thus $v(x) = \operatorname{grad} f$.

Recall that a (not necessarily symmetric) Minkowski metric in $\mathbf{R}^n$ is given by a smooth quadratically convex hypersurface $S \subset \mathbf{R}^n$ which contains the origin. This hypersurface consists of Minkowski unit vectors. By definition, each vector $x \in S$ is Minkowski normal to the hyperplane $T_x S$.

LEMMA 2.4. *Given a smooth Minkowski metric in* $\mathbf{R}^n$ *and a hypersurface* $M \subset \mathbf{R}^n$, *the field of Minkowski normals to* M *is exact.*

Proof. Identify vectors and covectors by the Euclidean structure; in particular, a covector, annihilating the tangent hyperplane T_xM, identifies with the normal vector $\nu(x)$. Let L be a positive homogeneous of degree 2 function on $\mathbf{R}^n - 0$ such that $S = L^{-1}(1)$. Then the Minkowski normal to M at $x \in M$ is given by the vector u, determined from the equation $L_u(u) = \nu(x)$. Therefore $n(x) = cu$ where the coefficient c is found from the condition $n(x)\,\nu(x) = 1$, that is, $cuL_u = 1$. By Euler's formula $uL_u = 2L$, thus $c = 1/2L$. Finally,

$$2n\,d\nu = -2\nu dn = -L_u d\Big(\frac{u}{L}\Big) = -\frac{L_u du}{L} + \frac{(uL_u)dL}{L^2} = -\frac{dL}{L} + \frac{2dL}{L} = d\log L,$$

and the result follows.

REMARKS. 1) Let M be a closed convex plane curve. Then a transverse line field ξ is exact if and only if there exists a parameterization $M(t)$ such that ξ is generated by the acceleration vectors $M''(t)$ – see [T2,3].

2) It is interesting to find all Finsler metrics in $\mathbf{R}^n$ such that for every hypersurface M the field of Finsler normals to M is exact.

The class of transverse line fields is not invariant under diffeomorphisms of $\mathbf{R}^n$. Surprisingly, the following result holds (generalizing to higher dimensions an observation from [T3]).

Theorem 2.5. *Let* ξ *be an exact transverse line field along* $M \subset \mathbf{R}^n$, *and* F *be a projective transformation of* $\mathbf{R}^n$ *whose domain contains* M. *Then the line field* $dF(\xi)$ *along* $F(M)$ *is also exact.*

Proof. Unfortunately the only proof I know is computational.

Use the same notation ν and n for the pair (M, ξ), and let $\bar\nu$ and $\bar n$ have the similar meaning for the pair $(F(M), dF(\xi))$. We know that $n\,d\nu$ is exact and want to show that $\bar n\,d\bar\nu$ is exact too.

Denote the matrix dF by A, and set $B = (A^*)^{-1}$, $f = |B(\nu)|$. Note that $A(u)\,B(v) = uv$ for every vector u and v.

We claim that $\bar\nu = B(\nu)/f$ and $\bar n = fA(n)$. Indeed, if u is tangent to M then $B(\nu)\,A(u) = \nu u = 0$. Therefore $B(\nu)$ is normal to $F(M)$. Likewise, $fA(n)\,B(\nu)/f = n\nu = 1$.

It follows that

$$\bar n\,d\bar\nu = fA(n)\left(\frac{dB(\nu)}{f} - \frac{B(\nu)df}{f^2}\right) = A(n)\,dB(\nu) - (n\nu)\frac{df}{f}\,.$$

The second term equals $d\log f$, and it remains to show that the first term is exact.

If $F = A$ is a linear transformation then

$$A(n)\, dB(\nu) = A(n)\, B(d\nu) = n\, d\nu\,,$$

and we are done. Every projective transformation is a composition of a linear one, a parallel translation and a fractional linear transformation

$$F(x) = \frac{x}{1+ax}$$

where a is a fixed vector. It remains to consider such an F.

In this case a direct computation yields:

$$A(u) = \frac{u}{1+ax} - \frac{(au)x}{(1+ax)^2}\,, \quad B(u) = (1+ax)\big(u + (xu)a\big)\,.$$

It follows that

$$A(n)\, dB(\nu) = \big(n\ (\nu + (x\nu)a)\big)\frac{d(1+ax)}{1+ax} - (an)\big(x\ (\nu + (x\nu)a)\big)\frac{d(1+ax)}{(1+ax)^2}$$
$$+\, n\big(d\nu + a\, d(x\nu)\big) - \frac{an}{1+ax}\big(x(d\nu + a\ d(x\nu))\big)\,.$$

Collect terms in the first two summands, taking into account that $n\nu = 1$, to obtain $d\log(1+ax)$. Similarly, the third and fourth terms give:

$$nd\nu + \frac{an}{1+ax}(\nu dx)\,.$$

The 1-form $nd\nu$ is exact by assumption, and νdx vanishes on M because ν is a normal vector field. The theorem is proved.

The rest of this section is devoted to an interpretation of the concept of exact transverse line fields in terms of contact geometry; our general reference is [AG].

Start with the following situation. Let L be a Legendrian submanifold in a contact manifold, and let η be a line field along L, transverse to the contact structure. Assume that the contact structure is coorientable near L, that is, can be given by a contact 1-form λ_0. Does there exist another contact form λ (for the same contact structure) such that η is its characteristic field, i.e. $\eta = \operatorname{Ker} d\lambda$? If this is the case we call η a characteristic line field.

The next lemma provides an answer (obtained in [T3] for dim $L = 1$).

LEMMA 2.6. *Let v be a vector field along L, generating η. The field η is characteristic if and only if the 1-form $i_v\, d\lambda_0/\lambda_0(v)$ is exact on L.*

Proof. Denote the form $i_v\, d\lambda_0/\lambda_0(v)$ by $\beta(\lambda_0)$; it clearly does not depend on the choice of v. If $\lambda = f\lambda_0$ is another contact form then

$$\beta(\lambda) = \frac{v(f)\lambda_0}{f\lambda_0(v)} - \frac{df}{f} + \beta(\lambda_0)\,.$$

Since L is Legendrian the first term vanishes on L. Thus the cohomology class of β is independent of the choice of the contact form. If $\eta = \operatorname{Ker} d\lambda$ then $\beta(\lambda) = 0$. Conversely, if $\beta(\lambda_0)$ is exact then there exists λ such that $\beta(\lambda) = 0$. Then $\eta = \operatorname{Ker} d\lambda$.

Consider the space of cooriented contact elements in $\mathbf{R}^n$ with its canonical contact structure. A hypersurface $M \subset \mathbf{R}^n$ determines a Legendrian submanifold $\tilde{M}$ which consists of tangent hyperplanes to M. Lift a transverse line field ξ along M to a line field $\tilde{\xi}$ along $\tilde{M}$ by parallel translating each tangent hyperplane T_xM along the line $\xi(x)$. Clearly, the field $\tilde{\xi}$ is transverse to the contact structure.

Theorem 2.7. *A transverse line field ξ along M is exact if and only if the field $\tilde{\xi}$ along $\tilde{M}$ is characteristic.*

Proof. The space of cooriented contact elements in $\mathbf{R}^n$ is contactomorphic to the space of 1-jets of functions on S^{n-1} with its canonical contact structure determined by the form $\lambda_0 = du - p\,dq$; here q is a point of the sphere, u is the value of a function and p is the differential. The contactomorphism acts as follows: given a contact element at point x, let q be its coorienting unit normal, $u = qx$ and $p = x - uq$.

As before, let ν be the unit normal vector field along M and n be the vector field, generating ξ, such that $n\nu = 1$. In (q, p, u) coordinates the line field $\tilde{\xi}$ is generated by the vector field

$$v = \frac{\partial}{\partial u} - (n - \nu)\frac{\partial}{\partial p}\,.$$

Therefore $\lambda_0(v) = 1$ and $i_v\, d\lambda_0 = -(n - \nu)dq$. Since ν is the unit normal field one has $q = \nu$ on $\tilde{M}$, and $\nu\, d\nu = 0$. Thus $\beta(\lambda_0) = -n\, d\nu$, and the result follows from Lemma 2.6.

The vector field v, generating $\operatorname{Ker} d\lambda$ and normalized so that $\lambda(v) = 1$, is called the Reeb field. The flow of a Reeb field preserves the contact structure, and, in particular, takes Legendrian manifolds to Legendrian ones. A Legendrian manifold of the space of cooriented contact elements in $\mathbf{R}^n$ consists of the tangent hyperplanes to a hypersurface (possibly, singular). This suggests another characterization of the exact transverse line fields.

Let n and ν have the same meaning as before. Given a function ϕ on M, translate every point $x \in M$ through the vector $\phi(x)n(x)$, and denote the resulting hypersurface by $M(\phi)$,

$$M(\phi) = \left\{X(x) \;\middle|\; X(x) = x + \phi(x)n(x)\,,\ x \in M\right\}.$$

Theorem 2.8. *Let the field $n(x)$ be exact: $n\,d\nu = df$ on M. Then the tangent hyperplanes $T_{X(x)}M(e^f)$ and T_xM are parallel for every $x \in M$. Conversely, if for some positive function ϕ, the hyperplanes $T_{X(x)}M(e^f)$ and T_xM are parallel for every $x \in M$ then the line field, generated by $n(x)$, is exact.*

Proof. The 1-form νdx vanishes on M; one wants to show that the form $\nu\,dX(x)$ vanishes on $M(e^f)$. One has:

$$dX = dx + e^f(n\,df + dn)\,,$$

therefore

$$\nu\,dX = \nu\,dx + e^f(n\nu\,df + \nu\,dn) = e^f(df - n\,d\nu) = 0\,.$$

Conversely, if $X = x + \phi n$ then

$$\nu\,dX = \nu\,dx + \nu n\,d\phi + \phi\nu\,dn = d\phi - \phi n\,d\nu$$

on M. If this 1-form vanishes then $nd\nu = d\log\phi$.

Clearly, Theorem 2.8 implies Lemma 2.4.

3 Invariant Symplectic Form

Let $n(x)$ be a transverse, not necessarily exact, vector field along the unit sphere $M \subset \mathbf{R}^n$ such that $n(x)\,x = 1$ for all $x \in M$. Consider the projective billiard transformation T of the space of oriented lines, intersecting the interior of the unit ball. Such a line is characterized by a pair $(x,y) \in M \times M$, $x \neq y$, its first and second intersection points with M. Thus T is a transformation of $M \times M - Diag$, namely $T(x,y) = (y,z)$.

Lemma 3.1. *If $T(x,y) = (y,z)$ then*

$$\frac{y-x}{1-xy} - \frac{z-y}{1-yz} = 2n(y)\,.$$

Proof. Let π be the 2-plane generated by the vectors $y-x$ and $z-y$. There exist reals a and b such that the line $\pi \cap T_yM$ is generated by the vector $w = a(y-x) + b(z-y)$. Since $wy = 0$ one has $a(1-xy) = b(1-yz)$. Thus one may take

$$w = \frac{y-x}{1-xy} + \frac{z-y}{1-yz}\,.$$

Note that, given vectors u and v, the lines generated by the vectors $u, v, u+v$ and $u-v$ form a harmonic quadruple. Taking $u = (y-x)/(1-xy)$ and $v = (z-y)/(1-yz)$ it follows that

$$cn(y) = \frac{y-x}{1-xy} - \frac{z-y}{1-yz}$$

for some real c. To find c take the scalar product with y:

$$c = cn(y)\ y = \frac{1-xy}{1-xy} - \frac{zy-1}{1-yz} = 2\,.$$

The lemma is proved.

Now let n be an exact transverse field.

Theorem 3.2. *The 2-form*

$$\omega = \frac{dx \wedge dy}{1-xy} + \frac{(y\,dx) \wedge (x\,dy)}{(1-xy)^2}$$

is a T-invariant symplectic form on $M \times M - Diag$.

Proof. Differentiate the equation from Lemma 3.1:

$$\frac{dy-dx}{1-xy} + \frac{(y-x)\ d(xy)}{(1-xy)^2} = \frac{dz-dy}{1-yz} + \frac{(z-y)d(yz)}{(1-yz)^2} + 2dn(y)\,,$$

and wedge multiply by dy:

$$\frac{dx{\wedge}dy}{1{-}xy}+\frac{((y{-}x)dy){\wedge}d(xy)}{(1{-}xy)^2} = \frac{dy{\wedge}dz}{1{-}yz}+\frac{((z{-}y)dy){\wedge}d(yz)}{(1{-}yz)^2}+2dy \wedge dn(y)\,.$$

Since $dy \wedge dn(y) = 0$ and $y\,dy = 0$ on M one has:

$$\frac{dx \wedge dy}{1-xy} + \frac{(y\,dx) \wedge (x\,dy)}{(1-xy)^2} = \frac{dy \wedge dz}{1-yz} + \frac{(z\,dy) \wedge (y\,dz)}{(1-yz)^2}\,,$$

that is, $T^*\omega = \omega$.

Given an oriented line xy with $x, y \in M$, let

$$p = \frac{x+y}{2}\ , \quad q = \frac{y-x}{|y-x|}\,.$$

Then $q^2 = 1$, $pq = 0$, $p^2 < 1$ and

$$x = p - (1-p^2)^{1/2} q\ , \quad y = p + (1-p^2)^{1/2} q\,.$$

In these new coordinates

$$\omega = \frac{dp \wedge dq}{(1-p^2)^{1/2}} + \frac{(p\,dp) \wedge (p\,dq)}{(1-p^2)^{3/2}} = d\left(\frac{p\,dq}{(1-p^2)^{1/2}}\right)\,.$$

The (p,q)-space is identified with the cotangent bundle T^*S^{n-1} where q is the coordinate and $(1-p^2)^{-1/2}p$ the momentum variables. Therefore the 1-form $(1-p^2)^{-1/2}p\,dq$ is the Liouville form on T^*S^{n-1}, and ω is a symplectic form.

REMARKS. 1) The following formulas hold: $\omega = d\lambda$ with $\lambda = -(1 - xy)^{-1} y\, dx$, and $\omega = d_y d_x \log(1 - xy)$.

2) If M is a circle then $dy \wedge dn(y) = 0$ on M for every transverse field n, not necessarily an exact one. Therefore the projective billiard transformation in a 2-dimensional disc always has an invariant symplectic form, the fact proved in [T2].

3) The form ω gives an infinite symplectic volume to the space of rays, intersecting the interior of the unit ball.

Recall the construction of the symplectic structure on the space of geodesics of a Riemannian manifold N (assuming the space of geodesics is a manifold – see [AG]). A metric determines the unit covector hypersurface $U \subset T^*N$. The restriction of the canonical symplectic form Ω from T^*N to U has a 1-dimensional kernel E. The integral curves of the distribution E identify with the geodesics on N. Thus Ω descends to a symplectic structure on the space of geodesics.

Theorem 3.3. *The 2-form ω from Theorem 3.2 coincides, up to the sign, with the symplectic form on the space of geodesics in the hyperbolic geometry inside the unit ball in $\mathbf{R}^n$.*

Proof. First recall the construction of the hyperbolic metric in the unit ball. Let H be the upper sheet of the hyperboloid $x^2 - y^2 = -1$ in $\mathbf{R}^n_x \times \mathbf{R}^1_y$ with the Lorentz quadratic form $dx^2 - dy^2$. The restriction of the Lorentz metric to H is a metric of constant negative curvature.

Project H from the origin to the hyperplane $y = 1$. Let p be the Euclidean coordinate in this hyperplane. The projection is given by the formula

$$x = \frac{p}{(1 - p^2)^{1/2}}\ , \quad y = \frac{1}{(1 - p^2)^{1/2}}\ ,$$

and the image of H is the open unit ball $p^2 < 1$. The hyperbolic metric g in the unit ball is given then by the formula:

$$g(u, v) = \frac{uv}{1 - p^2} + \frac{(up)(vp)}{(1 - p^2)^2}\ ,$$

where u and v are tangent vectors at p, and the scalar product is the Euclidean one. Thus

$$g_{ij} = \frac{\delta_{ij}}{1 - p^2} + \frac{p_i p_j}{(1 - p^2)^2}\ .$$

The geodesics of this metric are straight lines.

Identify the tangent and cotangent bundles by the metric: if ξ is the covector, corresponding to a tangent vector u, then $\xi_i = \sum_j g_{ij} u_j$. The Liouville 1-form $\xi\, dp$ becomes the following 1-form on the tangent bundle:

$$\alpha = \frac{u\, dp}{1-p^2} + \frac{(pu)(p\, dp)}{(1-p^2)^2}\,.$$

The space of geodesics identifies with the submanifold of the tangent bundle:

$$V = \big\{(p,u) \mid g(u,u) = 1\,,\ up = 0\big\}\,.$$

If q is the Euclidean unit vector proportional to u then $u = (1-p^2)^{1/2} q$ and $pq = 0$. It follows that

$$\alpha|_V = \frac{u\, dp}{1-p^2} = \frac{q\, dp}{(1-p^2)^{1/2}} = -\frac{p\, dq}{(1-p^2)^{1/2}}\,.$$

Therefore the symplectic form on geodesics is

$$-d\left(\frac{p\, dq}{(1-p^2)^{1/2}}\right)\,,$$

which coincides, up to the sign, with the formula for ω given in the proof of Theorem 3.2.

Consider the (usual) billiard transformation T inside an ellipsoid $E \subset \mathbf{R}^n$. As every billiard transformation, T preserves the canonical symplectic structure ω on the space of oriented lines in $\mathbf{R}^n$, associated with the Euclidean metric (see, e.g. [T1]). The transformation T is integrable, the fact that goes back to Jacobi (see, e.g. [MosV]).

A linear transformation that takes E to a sphere transforms T to a projective billiard transformation inside a ball. By Theorem 2.5 or Lemma 2.4 the corresponding transverse line field along the sphere is exact. By Theorem 3.2 this projective billiard transformation has an invariant symplectic form. Returning back to E and taking Theorem 3.3 into account, one obtains the following result.

Theorem 3.4. *Let Ω be the symplectic form on the space of oriented lines, intersecting an ellipsoid E, which is associated with the hyperbolic metric inside E (the Beltrami-Klein model). Then Ω is invariant under the billiard transformation inside the ellipsoid.*

Two symplectic structures on the same manifold are related by a tensor field J of type (1,1): $\Omega(u,v) = \omega(Ju,v)$ for every tangent vectors u and v. Each symplectic structure determines a Poisson structure, thought of as a bivector field. The two symplectic structures are called Poisson compatible if the spectrum of J is simple and real, and the sum of the two

Poisson structures is again a Poisson structure. If this is the case then every transformation that preserves both symplectic structures is integrable: the eigenvalues of J are integrals in involution with respect to both structures (this is a discrete version of the "bihamiltonian formalism"; see [M], [BMoTu]).

It is very tempting to expect the Poisson compatibility of the forms ω and Ω on the space of oriented lines, associated with the Euclidean and hyperbolic metric inside an ellipsoid. However a direct computation which we do not reproduce here shows that this is not the case.

4 Generating Function and Periodic Orbits

Let M be a manifold and T a transformation of $M \times M$ such that $T(x,y) = (y,z)$. A generating function for T is a function $L(x,y)$ on $M \times M$ such that

$$d_y\big(L(x,y) + L(y,z)\big) = 0 \ \text{ iff } \ T(x,y) = (y,z)\,.$$

If T is the usual billiard transformation inside a convex hypersurface M, one may take $dist(x,y)$ as a generating function.

Let M be the unit sphere with an exact transverse vector field n such that $n(x)\,x = 1$. By Lemma 2.3 there exists a function f on M such that $n(x) = x + \operatorname{grad} f$. Extend f to $\mathbf{R}^n - 0$ as a homogeneous function of degree 0; the gradient of f at every $x \in M$ coincides with the gradient of the restriction of f to M.

Let T be the corresponding projective billiard transformation of $M \times M - Diag$.

LEMMA 4.1. *The function*

$$L(x,y) = 2f(y) - \log(1 - xy)$$

on $M \times M - Diag$ is a generating function for T.

Proof. If $T(x,y) = (y,z)$ then, by Lemma 3.1,

$$\frac{y - x}{1 - xy} - \frac{z - y}{1 - yz} = 2n(y)\,.$$

Scalar multiply by dy and take into account that $y\,dy = 0$ on M:

$$2df(y) + \frac{x\,dy}{1 - xy} + \frac{z\,dy}{1 - yz} = 0\,,$$

that is, $d_y(L(x,y) + L(y,z)) = 0$.

Conversely, let

$$2df(y) + \frac{x\,dy}{1 - xy} + \frac{z\,dy}{1 - yz} = 0$$

on M. Then the left-hand side is proportional to $y\,dy$, and one has

$$2f_y + \frac{x}{1-xy} + \frac{z}{1-yz} = cy\,.$$

To find the coefficient c, scalar multiply by y keeping in mind that $yf_y = 0$:

$$c = \frac{xy}{1-xy} + \frac{zy}{1-yz}\,.$$

Thus

$$2n(y) = 2f_y + 2y = \frac{(xy)y - x}{1-xy} + \frac{(zy)y - z}{1-yz} + 2y = \frac{y-x}{1-xy} - \frac{z-y}{1-yz}\,,$$

and the result follows from Lemma 3.1.

REMARK. If a transformation has a generating function $L(x,y)$ then it preserves the 2-form $d_x d_y L(x,y)$ – compare with Remark 1 in section 3.

One applies generating functions to study periodic orbits of the transformation: T-periodic orbits $(x_1,...,x_n)$, $x_i \in M$ correspond to critical points of the function

$$L(x_1,x_2) + L(x_2,x_3) + ... + L(x_n,x_1)$$

on $M \times ... \times M$ (n times).

Theorem 4.2. *The projective billiard transformation inside a ball, associated with an exact transverse line field along the boundary sphere, has n-periodic orbits for every $n \geq 2$.*

Proof. Let L be the function from Lemma 4.1. The function

$$L(x_1,x_2) + L(x_2,x_3) + ... + L(x_n,x_1)$$

is defined off the set $D = \bigcup_{i=1}^{n} \{x_i = x_{i+1}\} \subset M \times ... \times M$ and goes to infinity as its argument tends to D. It follows that this function has a minimum on $M \times ... \times M - D$, and this provides a periodic orbit (which may be a multiple orbit if n is not prime).

Theorem 4.2 implies interesting geometrical properties of exact transverse fields along a sphere.

COROLLARY 4.3. *Given an exact transverse line field ξ along the sphere M, there exist points $x, y \in M$ such that the lines $\xi(x), \xi(y)$ and (xy) coincide. There also exist points $x, y, z \in M$ such that the lines $\xi(x), \xi(y)$ and $\xi(z)$ intersect at one point.*

Proof. If the projective billiard reflection at point $x \in M$ preserves a (nonoriented) line then this line is $\xi(x)$. Thus the existence of 2-periodic orbits implies the first claim.

Let $\{x, y, z\}$ be a 3-periodic orbit; denote the plane through x, y, z by π.

The lines $\xi(x), \xi(y)$ and $\xi(z)$ lie in π. There exists a projective transformation F of π, preserving the circle $\pi \cap M$, that sends the points x, y, z to the vertices of an equilateral triangle. Then the lines $\xi(x), \xi(y)$ and $\xi(z)$ are sent by F to the medians of this triangle. Therefore these lines intersect at one point.

Remark. I do not know whether an analog of Corollary 4.3 holds for every exact transverse line field along an arbitrary closed convex hypersurface.

5 Continuous Analog of Projective Billiards: Projective Geodesics

Let M be a hypersurface in $\mathbf{R}^n$ equipped with a transverse line field ξ. If a line makes a small angle with M then its image under the projective billiard transformation again makes a small angle with M. As the angle goes to zero one expects to obtain a continuous motion on M.

We define this motion as follows. Let $x(t) \subset M$ be a parameterized curve. Denote by $x_{\pm}(t, \epsilon)$ the points, obtained from $x(t \pm \epsilon)$ by truncating the Taylor expansion in ϵ at terms of order 2:

$$x_{\pm}(t, \epsilon) = x(t) \pm \epsilon x'(t) + \frac{\epsilon^2}{2} x''(t)\,;$$

here prime denotes d/dt and ϵ is sufficiently small. Assume that the vectors x' and x'' are not colinear for all t.

Definition. A curve $x(t) \subset M$ is called a projective geodesic, associated with a transverse line field ξ along M, if, for every t, the corresponding projective billiard transformation takes the oriented line $(x_-(t, \epsilon), x(t))$ to $(x(t), x_+(t, \epsilon))$.

Lemma 5.1. *A curve $x(t)$ is a projective geodesic if and only if, for all t, the acceleration vector $x''(t)$ is colinear with $\xi(x(t))$.*

Proof. Let $x(t)$ be a projective geodesic. The first condition in the definition of the projective billiard transformation implies that $\xi(x(t))$ lies in the plane, spanned by $x'(t)$ and $x''(t)$. Note that the vectors

$$-x'(t) + \frac{\epsilon}{2} x''(t)\,, \quad x'(t) + \frac{\epsilon}{2} x''(t)\,, \quad x''(t) \;\text{ and }\; x'(t)$$

generate a harmonic quadruple. Thus $\xi(x(t))$ and $x''(t)$ must be colinear. The converse holds by the same argument, and we are done.

One uses this lemma to extend the definition of projective geodesics to the case when $x'(t)$ and $x''(t)$ may be colinear for some t.

If the transverse field consists of the Euclidean normals to M then a projective geodesic is the usual one. A similar result holds for any metric g in a domain, containing M, if the transverse field consists of g-normals to M.

A transverse field ξ along M determines a projection

$$p : T_x\mathbf{R}^n \to T_xM \ , \quad x \in M \ ; \quad Ker\, p = \xi(x) \, .$$

A connection on M arises: to parallel translate a vector along a curve on M one parallel translates it in the ambient space and then projects to TM. Projective geodesics are the geodesics of this connection.

Remark. It would be interesting to investigate how the exactness of a transverse line field reflects in the properties of the corresponding connection.

The next lemma shows that the class of nonparameterized projective geodesics is invariant under the projective transformations of $\mathbf{R}^n$.

Lemma 5.2. *Let F be a projective transformation of a domain, containing M, and $x(t)$ be a projective geodesic on M, associated with a transverse line field ξ. Then there exists a reparameterization $t(\tau)$ of the curve $x(t)$ such that the curve $F(x(\tau))$ is a projective geodesic on $F(M)$, associated with the field $dF(\xi)$.*

Proof. A projective transformation takes the 2-dimensional osculating plane of a curve to the 2-dimensional osculating plane of its image. That is, $F''(x(t))$ lies in the plane, spanned by the vectors $dF(x'(t))$ and $dF(x''(t))$.

Let τ be another parameter; denote $d/d\tau$ by dot. One has;

$$\dot{F}\big(x(\tau)\big) = \dot{t}F'\big(x(t)\big) \, ; \quad \ddot{F}\big(x(\tau)\big) = \dot{t}^2F''\big(x(t)\big) + \ddot{t}F'\big(x(t)\big) \, .$$

It follows that one may choose τ so that $\ddot{F}(x(\tau))$ is colinear with $dF(x'')$, and therefore, with $dF(\xi)$ (a computation behind this claim is left to the reader). Then, by Lemma 5.1, the curve $F(x(\tau))$ is a projective geodesic, associated with the field $dF(\xi)$.

As before, let ν be the unit normal vector field along M, and n be the vector field, generating ξ, such that $\nu n = 1$. Denote by B the second quadratic form of M, and let $v(x) = n(x) - \nu(x)$.

Lemma 5.3. *A projective geodesic $x(t)$ on M satisfies the differential equation,*

$$x''(t) = -B\big(x'(t)\big)\, n\big(x(t)\big) \, .$$

Proof. By Lemma 5.1, $x'' = cn(x)$. To find the coefficient c differentiate the equation $\nu(x)\,x' = 0$:

$$\nu'(x)\,x' + c\nu(x)\,n(x) = 0\,.$$

The first term is $B(x')$, hence $c = -B(x')$. The lemma is proved.

It follows that the geodesic acceleration of a projective geodesic $x(t)$, i.e. the orthogonal projection of x'' to T_xM, equals $-B(x')\ v(x)$. One arrives at a flow on TM which we call the projective geodesic flow,

$$x' = u\,, \quad u' = -B(u)\,v(x)\,,$$

where $x \in M$, $u \in T_xM$. For example, if M is the unit sphere then $u' = -(u^2)\,v(x)$.

The projective geodesic flow is an interesting object of study. An intriguing question is for which pairs (M, ξ) there exists a Lagrangian whose extremals are the projective geodesics. One conjectures this to be the case for the exact transverse fields along a sphere.

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