REMARKS ON THE GEOMETRY OF EXACT TRANSVERSE LINE FIELDS

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To D. Fuchs with affection

1. Introduction

Exact transverse line fields along smooth hypersurfaces in linear or projective space is an interesting object of study closely related to projective, Finsler and symplectic geometries. Introduced in the study of a dynamical system, the so-called projective billiard ([3] – [5]), exact transverse line fields deserve attention on their own.

Start with a definition.

Let M be a smooth coorientable hypersurface in an affine space V and ξ be a smooth transverse line field along M. Choose a non-vanishing vector field v along ξ and let p be the conormal covector field on M normalized by the condition: $pv = 1$. The field ξ is called *exact* if the differential 1-form vdp is exact on M.

Here and below we use the vector notation: v is a vector-valued function, p is a covector-valued one, dp is a covector-valued 1-form, so vdp is a 1-form on M. The above definition does not depend on the choice of v: if $v_1 = v/f$ is another section of ξ where f is a function then $p_1 = fp$ and $v_1 dp_1 = vdp + d\ln f$. If M is a curve then exactness simply means that $\int_M vdp = 0$; if dim $M \geq 2$ then exactness is also a meaningful local condition.

Remark. The next equivalent definition of exactness may clarify the notion (suggested by V. Arnold and implicit in [5]). Given a hypersurface $M \subset V$ equipped with a transverse line field ξ, consider the following map ϕ from the conormal bundle of M (with the zero section deleted) to T^*V:

$$\phi(x,p) = (q,p); \quad \text{where} \;\; q \in \xi(x), \;\; \text{and} \;\; p(q-x) = 1.$$

In words: every conormal covector is parallel translated along the transverse line that passes through its foot point, the magnitude of translation being inverse proportional to that of the covector. Then the field ξ is exact if and only if the image of ϕ is an exact Lagrangian submanifold in T^*V (possibly, singular).

As a matter of motivation and to put the exposition into perspective we list below some examples and properties of exact transverse line fields from [3] – [8].

(o) The simplest example of an exact field is the field of the Euclidean normals to a hypersurface. Let n be the unit normal vector field. Identifying vectors and covectors by the Euclidean structure, one has: $p = n$, and $ndn = 0$ since $n^2 = 1$. Exact transverse line fields generalize Euclidean normals and enjoy many geometrical properties of the latter.

Another simple example: if M is star-shaped with respect to a point x then the field of lines through x is exact.

(i) Let g be a Riemannian metric of constant positive or negative curvature in a domain $D \subset V$ whose geodesics are straight lines. Then the field of g-normals to every hypersurface in D is exact.

(ii) If V is a Minkowski space with a smooth quadratically convex but not necessarily symmetric unit sphere then the field of Minkowski normals to every cooriented hypersurface is exact.

(iii) Let $M \subset V$ be a strictly convex hypersurface. Then the affine normals to M constitute an exact transverse line field.

(iv) Although defined in linear terms, exactness is a projective property. Namely, let ξ be an exact transverse line field along $M \subset V$, and F be a projective transformation of V whose domain contains M. Then the line field $DF(\xi)$ along $F(M)$ is also exact.

(v) Let $M \to \mathbf{RP}^n$ be an immersed quadratically non-degenerate closed hypersurface. Then for every exact transverse line field ξ along M and every point x in $\mathbf{RP}^n$ the number of lines from ξ passing through x is not less than the least number of critical points of a smooth function on M.

(vi) Let ξ be an exact transverse line field along an ellipsoid $E \subset \mathbf{R}^n$. Identifying the interior of the ellipsoid with the hyperbolic space $\mathbf{H}^n$ (Klein's model), one has an imbedding of E to the space of oriented lines in $\mathbf{H}^n$: to a point $x \in E$ there corresponds the line $\xi(x)$, oriented outward. Let L be the image of this map. Then L is a Lagrangian submanifold of the space of oriented lines in $\mathbf{H}^n$ with its canonical symplectic structure, associated with the hyperbolic metric, and if $n \geq 3$ this condition is equivalent to exactness.

Exact transverse line fields along the ellipsoid were used in [8] to provide a proof of the complete integrability of the geodesic flow therein.

(vii) Let γ be a smooth strictly convex closed plane curve. A smooth transverse line field ξ along γ is exact if and only if ξ is generated by the acceleration vectors $\gamma''(t)$ for some smooth parameterization $\gamma(t)$.

The content of the paper is as follows. In Section 2 we associate a differential 1-form to a pair of transverse line fields along a hypersurface in the projective space. This construction is given in purely projective terms, and it sheds new light on the above property (iv).

In Section 3 the notion of exactness is extended to transverse line fields along convex polyhedral hypersurfaces.

Given a transverse line field ξ along a hypersurface M, a ξ*-diameter* is a chord of M whose direction coincides with the directions of ξ at both end points. We prove in Section 4 that an exact transverse line field ξ along an $(n-1)$-dimensional sphere has at least n ξ-diameters. This is analogous to the well known fact that a closed convex hypersurface in n-dimensional Euclidean space has at least n diameters.

Given a transverse line field ξ along a hypersurface M, a *focal point* on a line from ξ is its intersection with an infinitesimally close line from the field (intersection is understood in the projective sense which includes parallel lines). We prove in Section 4 that if ξ is an exact field along an n-dimensinal sphere then there are exactly n focal points on every line from ξ (counted with multiplicities).

Section 5 contains three examples: a smooth closed plane curve with an exact transverse field whose lines do not cover the whole plane; a strictly convex closed plane curve with an exact transverse field ξ, free from ξ-diameters; and an exact transverse field along a quadratically non-degenerate surface free from focal points. These examples are in contrast with the above property (v) and the results of Section 4 . Similar examples can be constructed in any dimension.

In the last Section 6 we give a local classification of smooth Finsler metrics such that the field of Finsler normals to every smooth hypersurface is exact.

2. A DIFFERENTIAL 1-FORM FROM A PAIR OF TRANSVERSE LINE FIELDS

We start with the case of a smooth plane curve. Let ξ and η be two smooth transverse line fields along a curve γ. We will construct a 1-form $D(\xi,\eta)$ on γ that, in a sense, measures the difference between ξ and η.

Let τ be a tangent vector to γ at point x. Include τ into a vector field tangent to γ, and let y be the time-ϵ image of x under the flow of this vector field. One has four lines through the point x: the tangent to

γ, the line xy, the lines $\xi(x)$ and $\eta(x)$. Denote by $C(\epsilon)$ the cross-ratio of these lines. Since the first two lines are ϵ-close, $C(\epsilon)$ is ϵ-close to 1. We define the value of $D(\xi,\eta)$ at τ to be $\lim_{\epsilon\to 0}\ln C(\epsilon)$.

Let p be a non-vanishing conormal field along γ, and let u and v be the sections of ξ and η, normalized so that $pu = pv = 1$. The next result relates the form $D(\xi,\eta)$ with the forms udp and vdp; we also give an explicit formula for $D(\xi,\eta)$ in terms of an arbitrary parameterization of the curve.

Lemma 2.1. *$D(\xi,\eta) = vdp - udp$.*

Proof. Let $\gamma(t)$ be a parameterization such that $\gamma(0) = x$ and $\gamma'(0) = \tau$. One has: $y = \gamma(\epsilon)$ and $y - x = \epsilon\tau + \epsilon^2\gamma''(0)/2 + O(\epsilon^3)$.

Denote by $[\ ,\ ]$ the cross-product of vectors in the plane. The cross-ratio of the four lines equals

$$\frac{[y-x,v][\tau,u]}{[y-x,u][\tau,v]} = 1 + \epsilon\Big(\frac{[\gamma''(0),v]}{[\tau,v]} - \frac{[\gamma''(0),u]}{[\tau,u]}\Big) + O(\epsilon^2).$$

Thus

$$D(\xi,\eta) = \Big(\frac{[\gamma'',v]}{[\gamma',v]} - \frac{[\gamma'',u]}{[\gamma',u]}\Big)dt.$$

In particular, this expression does not depend on the parameterization and does not change under the projective transformations of the plane.

Return to the 1-form $vdp - udp$; it is easy to see that this form does not depend on the choice of the conormal field p. Identify vectors and covectors by the area form $[\ ,\ ]$. In particular, $p = \gamma'$ is a non-vanishing conormal field along γ. Let v be a section of ξ and let $v_1 = v/[\gamma', v]$. Then $pv_1 = 1$ and

$$v_1 dp = \frac{[\gamma'',v]}{[\gamma',v]}dt.$$

The result follows. $\square$

Remark. A consequence of the proof is the next criterium: a transverse field is exact if and only if

$$\int_\gamma \frac{[\gamma'',v]}{[\gamma',v]}dt = 0$$

for some (and then every) parameterization of the curve γ – compare with [3, 7].

The construction of the form $D(\xi,\eta)$ in the multi-dimensional case is a modification of that in the plane. Replace the curve by a hypersurface $M \subset \mathbf{RP}^n$ and let x, τ, ξ and η have the same meaning as before.

Denote by $y_\pm$ the time-($\pm\epsilon$) images of x under the flow of a vector field whose value at x is τ.

Start with a general consideration. Given a point $P \in \mathbf{RP}^n$, let π_1, π_2 be a pair of hyperplanes and l_1, l_2 a pair of lines containing P. Assign a number C to this data as follows. If $\pi_1 = \pi_2$ then $C = 0$; otherwise let $\nu = \pi_1 \cap \pi_2$. Factorizing $T_P\mathbf{RP}^n$ by ν one obtains a plane and four concurrent lines in it, the projections of π_1, π_2, l_1, l_2. Let C be the cross-ratio of these lines.

Apply this construction to $\pi_1 = T_{y_-}M, \pi_2 = T_{y_+}M, l_1 = \xi(x), l_2 = \eta(x)$ and $P = x$. There is a complication in that x may not lie in $\nu = T_{y_-}M \cap T_{y_+}M$. If this is the case, choose a point $z \in \nu$ which is ϵ^2-close to x and replace the lines l_1 and l_2 by two lines through z which are ϵ^2-close to $\xi(x)$ and $\eta(x)$ (e.g., connect z with points on $\xi(x)$ and $\eta(x)$ at distance of order 1 from z).

Then the above construction provides a cross-ratio $C(\epsilon)$. As before, $C(\epsilon)$ is ϵ-close to 1, and we define $D(\xi, \eta)(\tau) = \lim_{\epsilon\to 0} \ln C(\epsilon)$. This number does not depend on the choices made. The statement of Lemma 2.1 remains valid.

Lemma 2.2. *$D(\xi, \eta) = 2(vdp - udp)$.*

Proof. Identify a neighborhood of x with a domain in $\mathbf{R}^n$ and let n be the unit normal vector field to M. Let u and v be the sections of ξ and η normalized so that $nu = nv = 1$. We want to prove that $D(\xi, \eta) = 2(vdn - udn)$.

Denote by n' the directional derivative of the vector-valued function n along the vector τ at point x. Then $n(y_\pm) = n(x) \pm \epsilon n' + O(\epsilon^2)$. Since ν is perpendicular to $n(y_-)$ and to $n(y_+)$, the formula $\pi(w) = (wn(y_+),\ wn(y_-))$ gives a projection $\pi : \mathbf{R}^n \to \mathbf{R}^n/\nu$.

Denote $(n')^2$ by ϕ. Computing modulo ϵ^2, one has: $(n' \mp \epsilon\phi\ n) \in T_{y_\pm}M$. Therefore

$$\pi(T_{y_-}M) = \epsilon\phi\ (1, 0), \quad \pi(T_{y_+}M) = \epsilon\phi\ (0, 1),$$

$$\pi(\xi(x)) = (1 + \epsilon\ un', 1 - \epsilon\ un'), \quad \pi(\eta(x)) = (1 + \epsilon\ vn', 1 - \epsilon\ vn').$$

Computing the cross-ratio of the lines generated by these four vectors in terms of the cross-products yields:

$$C(\epsilon) = \frac{(1 + \epsilon\ vn')(1 - \epsilon\ un')}{(1 - \epsilon\ vn')(1 + \epsilon\ un')} + O(\epsilon^2) = 1 + 2\epsilon\ (vn' - un') + O(\epsilon^2),$$

and therefore $D(\xi, \eta)(\tau) = 2(vn' - un')$. To finish the proof it remains to notice that $vn' = (vdn)(\tau)$ and likewise for u. □

The construction of the form $D(\xi, \eta)$ implies the projective invariance of exactness (at least, locally). Indeed, let ξ be an exact transverse line field along M and let η be the field of lines through a fixed point which we assume to be transverse to M (locally this is always possible). Then η is exact and $D(\xi, \eta)$ is an exact 1-form on M. A projective transformation does not change $D(\xi, \eta)$ and takes η to another field of concurrent lines, which is exact again. Therefore the image of ξ is an exact field too.

Definition. Transverse line fields ξ and η are called *homologous* if the 1-form $D(\xi, \eta)$ is exact.

We finish this section with the following result.

Theorem 2.3. *If ξ and η are homologous transverse line fields along an immersed quadratically non-degenerate closed hypersurface M in $\mathbf{RP}^n$ then the number of points $x \in M$ such that $\xi(x) = \eta(x)$ is not less than the least number of critical points of a smooth function on M.*

Proof. Since $D(\xi, \eta)$ is the differential of a function on M the number of points at which this 1-form vanishes satisfies the desired estimate. One wants to show that if $D(\xi, \eta) = 0$ at x then $\xi(x) = \eta(x)$.

In local Euclidean coordinates $D(\xi, \eta) = (v - u)dn$ where, as before, n is the unit normal vector field to M. If $D(\xi, \eta)$ vanishes at point x then $((v - u)dn)(\tau) = 0$ for every tangent vector $\tau \in T_xM$. The expression $((v - u)dn)(\tau)$ is the value of the second quadratic form of M on the tangent vectors $v - u$ and τ. Since M is quadratically non-degenerate one concludes that $u(x) = v(x)$. $\square$

Remark. The non-degeneracy condition, in general, cannot be relaxed as the first example in Section 5 shows.

3. EXACT TRANSVERSE FIELDS ALONG CONVEX POLYHEDRAL HYPERSURFACES

We will define exact transverse line fields along polyhedral hypersurfaces approximating these hypersurfaces by smooth ones. Start with a definition.

Definition. A transverse line field along a convex closed polyhedral hypersurface P is given if to every point $x \in P$ there corresponds a line l through x that intersects the interior of P; the map $x \to l$ is continuous.

Consider first the case of a convex plane polygon P. Let γ be a smooth convex curve, approximating P, equipped with an exact transverse line field given by a vector field v normalized so that $vn = 1$ (where n is the unit normal vector field). The exactness condition is that $\int_\gamma vdn = 0$. The contribution to the integral of the arcs of γ, approximating the sides of P, is close to zero since dn vanishes along straight segments. Thus the integral localizes at the vertices. We compute the contribution of a vertex in the next lemma.

Lemma 3.1. *Let A be a vertex of P, and let the line l of the transverse field at A make the angles α and β with the sides. Then the contribution of this vertex to the integral equals* $\ln\,(\sin\beta/\sin\alpha)$*.*

Proof. Choose the origin at A and let l be the x-axis. The unit normal vector to P at A makes a turn through the angle $\pi-\alpha-\beta$ and is given by the formula

$$n(t)=(\cos t,\sin t),\ \ \alpha-\pi/2\le t\le \pi/2-\beta.$$

To satisfy the condition $vn=1$ one must have: $v(t)=(\cos t)^{-1}(1,0)$. Then $vdn=-\tan t\ dt$ which is easily integrated and yields the result. □

We arrive at the following definition. Given a transverse line field ξ along a convex plane k-gon P, denote by α_i and β_i the angles made by the lines of ξ at the i-th vertex of P with the adjacent sides.

Definition. The field ξ is called exact if $\Pi_1^k(\sin\alpha_i/\sin\beta_i)=1$.

Examples. A transverse line field along a triangle is exact if and only if its three lines at the vertices of the triangle are concurrent.

Each of the following three conditions imply exactness for every convex polygon: the lines at the vertices are the bisectors of the respective angles of the polygon; the lines at the vertices are concurrent; and the line at every vertex passes through the midpoint of the segment connecting the two adjacent vertices.

Remark. Let η be another transverse line field along a polygon whose lines at a vertex A make the angles $\bar\alpha$ and $\bar\beta$ with with the two adjacent sides. Then $(\sin\alpha\sin\bar\beta)/(\sin\beta\sin\bar\alpha)$ equals the cross-ratio of the four lies: the lines from the fields ξ and η at A and the adjacent sides of the polygon. The sum of logarithms of these cross-ratios over all vertices is the polygonal analog of the integral $\int_\gamma D(\xi,\eta)$ from the previous section.

The next result is a polygonal analog of property (vii), Section 1 .

Lemma 3.2. *Let $A_i,\ i=1,\dots,n$ be the vertices of a convex n-gon P. A transverse line field ξ along P is exact if and only if there exist positive reals t_i such that for every i the line $\xi(A_i)$ is generated by the vector $t_{i+1}(A_{i+1}-A_i)-t_i(A_i-A_{i-1})$.*

Proof. Set $z_i = (A_i - A_{i-1})$ and let $\xi(A_i)$ be generated by a vector v_i. Then $\Pi_1^n(\sin\alpha_i/\sin\beta_i) = \Pi_1^n([z_{i+1}, v_i]/[z_i, v_i])$. If $v_i = t_{i+1}z_{i+1} - t_i z_i$ then the latter product is $\Pi_1^n(t_i/t_{i+1}) = 1$.

Conversely, given the lines $\xi(A_i)$, there exist positive reals s_i such that these lines are generated by the vectors $v_i = s_{i+1}z_{i+1} - z_i$. If ξ is exact, the same argument as above implies that $\Pi_1^n s_i = 1$. Then one may set: $t_k = \Pi_1^k s_i$, and we are done. □

An exact transverse line field along a convex polygon can be arbitrarily well approximated by an exact transverse line field along a convex smooth curve. This implies the following result – compare with property (v), Section 1 .

Corollary 3.3. *Let ξ be an exact transverse line field along a convex polygon. For every point x in the plane there exist at least two lines from ξ that pass through x.*

The reader is invited to find a proof by elementary geometry!

Next consider a convex polyhedral hypersurface $P \subset \mathbf{R}^n$. Let v and n have the same meaning as before. Exactness means that for every oriented closed curve $\gamma \subset P$ one has: $\int_\gamma v dn = 0$. In particular, the integral does not change under a homotopy of the curve.

A generic curve intersects $(n-2)$-dimensional faces of P transversally, and the integral is localized at the intersection points. Let A be an intersection point of γ with an $(n-2)$-dimensional face F, and let l be the line of the transverse line field at A. Denote by α and β the angles made by the hyperplane spanned by F and l with the $(n-1)$-dimensional faces, adjacent to F; the order of the angles is prescribed by the orientation of the curve. A computation similar to the proof of Lemma 3.1 shows that the contribution to the integral at A is the same as in the statement of Lemma 3.1 .

Consider an oriented curve γ that crosses F at point A, makes a U turn and crosses back at a nearby point B. The contributions of A and B to the integral should cancel. This implies that the angles α and β at B are the same as those at A. Therefore the lines of an exact transverse line field along every $(n-2)$-dimensional face F of P must belong to a fixed hyperplane containing F.

More generally, if F is a k-dimensional face of P then there exists a $(k+1)$-dimensional affine space $S(F)$ containing F such that the lines $\xi(x),\ x \in F$ vary inside $S(F)$.

Indeed, arguing inductively, let F be the intersection of $(k+1)$-dimensional faces F_1 and F_2. For every point $A \in F$ the line $\xi(A)$ lies in $S(F_1) \cap S(F_2)$. These two $(k+2)$-dimensional spaces do not coincide and, since $n-k \geq 3$, their intersection is at most $(k+1)$-dimensional. Thus $S(F)=S(F_1)\cap S(F_2)$.

The above restriction on a transverse line field ξ is rather severe; it makes it possible to construct the spaces $S(F)$ from the values of ξ at the vertices of P. Namely, the following condition holds.

Exactness condition. If F is a k-dimensional face of P then all the lines from the field ξ at the vertices of F lie in a $(k+1)$-dimensional affine space $S(F)$ containing F, and the lines of ξ along F vary continuously inside $S(F)$.

In particular, if A and B are adjacent vertices of P then the lines $\xi(A)$ and $\xi(B)$ intersect (in $\mathbf{RP}^n$). Here is a generalization of this property.

Lemma 3.4. *Let ξ be a transverse line field along a convex polyhedral hypersurface P satisfying the exactness condition. If a face F of P is a k-dimensional simplex then all the lines from ξ at the vertices of F are concurrent.*

Proof. One has: dim $S(F)=k+1$. Let $G_1,...,G_{k+1}$ be the $(k-1)$-dimensional faces of F. Then the k-dimensional spaces $S(G_i)\subset S(F)$ intersect at a point, say, O. Every vertex A of F is the intersection of k faces, say, $G_1,...,G_k$. Then $\xi(A)=S(G_1)\cap ... \cap S(G_k)$; thus $\xi(A)$ passes through O. ☐

Returning to exactness, one has the following result.

Lemma 3.5. *Let ξ be a transverse line field along a convex polyhedral hypersurface P satisfying the exactness condition, and let γ be a generic oriented closed curve in P. Then $\int_\gamma v dn = 0$.*

Proof. Consider a generic homotopy γ_t of the curve to a point. We will show that the integral does not depend on t which implies the result. ☐

The contributions to the integral come from the intersections of the curve with $(n-2)$-dimensional faces of P. As long as these intersections remain transversal, the contributions do not change – this is guaranteed by the exactness condition. Generically, two singular events may happen: a pair of transverse intersection points appear or disappear, or γ_t may pass through an $(n-3)$-dimensional face – see Figure 1. In the former case the integral remains the same – this also follows from the discussion preceding the formulation of the exactness condition.

Consider the second move in Figure 1. Let F be the $(n-3)$-dimensional face involved and let $G_1, ..., G_k$ be the adjacent $(n-2)$-dimensional faces in the cyclic order around F. Denote by α_i and β_i the angles made by the hyperplane $S(G_i)$ with the two faces of P, the ones containing the pair G_i, G_{i-1} and the pair G_i, G_{i+1}, respectively. To show that the integral is invariant under the second move it suffices to prove that $\Pi_1^k(\sin\alpha_i/\sin\beta_i) = 1$.

Consider the 3-space V, orthogonal to F. In this space one has a polyhedral angle with the vertex $V \cap F$, edges $E_i = V \cap G_i$, and a line $l = V \cap S(F)$ inside the angle. Then α_i and β_i are equal to the angles made by the plane spanned by l and E_i with the faces of the polyhedral angle, adjacent to E_i. The identity $\Pi_1^k(\sin\alpha_i/\sin\beta_i) = 1$ is a fact of elementary geometry that can be proved as follows.

Choose a vector u along the line l, and let v_i be its projection on the edge E_i. Then the vector $u - v_i$ is perpendicular to E_i, and

$$\Pi(\sin\alpha_i/\sin\beta_i) = \Pi([u-v_i, v_i \times v_{i+1}]/[u-v_i, v_{i-1} \times v_i]) =$$

$$\Pi([u, v_i \times v_{i+1}]/[u, v_{i-1} \times v_i]) = 1.$$

The lemma is proved.

Thus the exactness condition provides an adequate definition of exact transverse line fields along convex polyhedral hypersurfaces. In particular, the statement of Corollary 3.3 applies to convex polyhedra in any dimension (and again, the reader is challenged to find an elementary geometry proof).

4. DIAMETERS AND FOCAL POINTS FOR EXACT TRANSVERSE LINE FIELDS ALONG A SPHERE

Start with the following result.

Theorem 4.1. *Let ξ be an exact transverse line field along an $(n-1)$-dimensional sphere (or an ellipsoid). Then the number of ξ-diameters is at least n.*

Proof. Using the Euclidean structure one identifies the position vector of a point of the unit sphere with a conormal covector. Let $v(x)$ be the section of ξ normalized so that $xv(x) = 1$. Then $vdx = df$ for some function f on $\mathbf{S}^{n-1}$.

Consider the function F on $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1} - \text{Diag}$ given by the formula: $F(x, y) = f(x) + f(y) - \ln(1 - xy)$. We claim that the critical points of this function correspond to the ξ-diameters.

Indeed, if

$$dF = df(x) + df(y) + \frac{xdy + ydx}{1 - xy} = 0$$

then

$$df(x) + \frac{ydx}{1 - xy} = 0 \quad \text{and} \quad df(y) + \frac{xdy}{1 - xy} = 0.$$

It follows that

$$v(x) + \frac{y}{1 - xy} = \lambda x \quad \text{and} \quad v(y) + \frac{x}{1 - xy} = \mu y$$

for some constants λ and μ. To find these constants dot-multiply the above equalities by x and y. One finds: $\lambda = \mu = 1/(1 - xy)$, therefore

$$v(x) = -v(y) = (x - y)/(1 - xy).$$

Thus xy is a ξ-diameter.

Let C be a big positive number. Then the set $\{F \le C\}$ is a compact manifold with boundary which is diffeomorphic to the set of oriented lines intersecting a unit ball. The latter is homotopically equivalent to the set of oriented lines through the center of the ball, that is, to the sphere $\mathbf{S}^{n-1}$. Consider the involution $\sigma(x, y) = (y, x)$ on $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$. The quotient manifold $\{F \le C\}/\sigma$ is homotopically equivalent to $\mathbf{RP}^{n-1}$.

The function F, being invariant under the involution σ, descends to a function $\bar{F}$ on the manifold $\{F \le C\}/\sigma$; its gradient is everywhere transverse to the boundary of this manifold (provided C is big enough).

Morse theory implies that the number of critical points of $\bar{F}$ is greater than the cup-length of $\mathbf{RP}^{n-1}$, that is, not less than n. $\square$

Remark. Property (vi) in Section 1 implies that the lines of an exact transverse field along the sphere (or an ellipsoid) are the hyperbolic normals to a closed hypersurface in the hyperbolic space. Thus the above theorem gives an estimate of the number of diameters in hyperbolic geometry; such a result is probably not new.

Next we study focal points of exact transverse line fields along the sphere. In view of the above remark the following Theorem 4.2 is not surprising: it just concerns focal points in the hyperbolic geometry. However it is interesting to work it out in terms of the sphere at infinity. The result below also follows from [1] where a similar fact is proved for the Finsler normals to a hypersurface provided the geodesics of the Finsler metric are straight lines.

Given a transverse line field ξ along a hypersurface M the focal points on a line $\xi(x),\ x \in M$ are its intersections with infinitesimally close lines from ξ. A tangent direction to M at x is called *principal* if, moving from x in this direction, the lines from ξ infinitesimally intersect $\xi(x)$ (this notion agrees with the familiar one from differential geometry if the field consists of the Euclidean normals).

Theorem 4.2. *Let ξ be an exact transverse line field along an n-dimensional sphere. Then there are exactly n focal points on every line from ξ, counted with multiplicities. If the focal points are distinct then the respective principal directions are orthogonal.*

Proof. Let ξ be given by the transverse vector field $v(x)$ normalized so that $xv = 1,\ x \in \mathbf{S}^n$. Then $vdx = df$ for a function f on the sphere. Extend f to the ambient space as a homogeneous function of degree 0. One has: $v(x) = x + \operatorname{grad} f$.

Let u be a tangent vector at x. Computing modulo ϵ^2, one wants the lines $\xi(x)$ and $\xi(x + \epsilon u)$ to intersect. This means that for some reals s and t one has:

$$x + \epsilon u + tv(x + \epsilon u) = x + (t + \epsilon s)v(x).$$

Denoting the directional derivative of v along u at x by $u(v)$ and equating the ϵ-terms yields: $u + tu(v) = sv$. Here and below we drop the argument to indicate the value of a function at point x, e.g, v is a shorthand for $v(x)$. Dot-multiply by x to find: $s = t\ u(v)x$. Since $u(v)x = (xdv)(u) = -(vdx)(u) = -u(f)$ one has: $s = -tu(f)$, and the equation on u to be solved reads:

$$u + t(u(v) + u(f)v) = 0.$$

This is an eigen-value problem for the linear map $A(u) = u(v) + u(f)v$ of the tangent space at x, and the claims of the theorem will follow once we show that A is self-adjoint.

Since $v(x) = x + \text{grad } f$ one has:

$$A(u) = u(x) + u(\text{grad } f) + u(f)x + u(f)\text{grad } f,$$

where the directional derivative $u(x)$ is just u. Let w be another tangent vector at x; then

$$A(u)w = uw + u(\text{grad } f)w + u(f)w(f).$$

The first and the third terms are clearly symmetric in u, w, and so is the second one since it equals the Hessian of the function f evaluated on the vectors u and w. $\square$

5. Three examples

The first is an example of a simple closed plane curve with an exact line field whose lines do not cover the whole plane.

Let $\gamma(t)$ be a closed curve parameterized by arc-length and let $v(t) = \gamma'(t) + \phi(t)J\gamma'(t)$ be a transverse vector field along the curve; here ϕ is a positive function and J is the linear operator of rotation through $\pi/2$. Then $[\gamma', v] = \phi$, $[\gamma'', v] = [kJ\gamma', v] = -k$ where k is the curvature. The exactness criterium reads:

$$\int_\gamma \frac{[\gamma'', v]}{[\gamma', v]} dt = \int_\gamma \frac{k}{\phi} dt = 0.$$

Start with a convex curve (say, sufficiently close to a circle) and the transverse field given by $\phi = 1$. This field is not exact and its lines do not cover the whole interior of the curve. Now make a small "dimple" in the curve – see Figure 2, and adjust the function ϕ as follows. Choose a smooth positive bump function f on the curve which is equal to 1 off the concave arc, assumes sufficiently great values on the concave arc and such that $\int_\gamma kf dt = 0$. Let $\phi = 1/f$.

The respective vector field v generates an exact line field, almost tangent to the curve along its negatively curved arc. The lines of this field still do not cover the whole interior of the curve.

The second example is a strictly convex closed plane curve with an exact transverse line field ξ, free from ξ-diameters. This example is obtained by a smooth approximation of an exact transverse field ξ along a triangle – see Figure 3.

Let ABC be an equilateral triangle, R, S, T be the midpoints of its sides and O the center of the triangle. The transverse line field ξ to be constructed will be invariant under the rotation through $2\pi/3$ about O. The lines of ξ at the vertices of the triangle will pass through O, therefore, according to Section 3, ξ will be exact.

Let $P = RT \cap BS$. The lines of ξ along the segment TB all pass through the point P, and on the segment BR they all pass through the point S. This completes the description of the field ξ. It is visually clear that there are no ξ-diameters; one can prove it as follows.

The field ξ determines a continuous selfmap f of the boundary of the triangle: $f(x)$ is the intersection point, other than x, of the line $\xi(x)$ with the boundary of the triangle. A ξ-diameter corresponds to a 2-periodic orbit of f. One has:

$$f^2 : TB \to RC \cup CS \to SA \cup AT \quad \text{and} \quad f^2 : BR \to S \to T.$$

Therefore f does not have 2-periodic orbits on $TB \cup BR$ and, by rotational symmetry, anywhere.

Our third example is that of an exact transverse line field along a quadratically non-degenerate surface in 3-space, free from focal points.

Consider a 2-parameter family of pair-wise skew lines in 3-space. An example of such a family is given by the rulings of a family of coaxial 1-sheeted hyperboloids. Parameterizing a line l_{xy} by its intersection point x, y with the horizontal plane one has a parametric equation: $l_{xy}(t) = (x + ty, y - tx, t)$. Infinitesimally close lines in this family do not intersect.

If $t = f(x, y)$ is a function of x, y then the locus of points $l_{xy}(t)$ is a surface M transverse to the family of lines. We want to describe the functions f such that this family of lines is exact along M. The answer is given by the next lemma; its proof is a straightforward but tedious computation, and we omit it.

Lemma 5.1. *The exactness holds if and only if the function* $\arctan f(x, y)$ *is harmonic.*

In particular, let $f = \tan(xy)$ in an ϵ-neighborhood of the origin. Then, modulo ϵ^3, the surface M is the quadric $z = xy$; in particular, M is quadratically non-degenerate for sufficiently small ϵ.

It follows from [1] that the constructed exact transverse line field is not the field of the Finsler normals to M for any Finsler metric whose geodesics are straight lines.

6. FINSLER METRICS WHOSE FIELDS OF NORMALS ALONG EVERY HYPERSURFACE ARE EXACT

In this section we classify smooth Finsler metrics in a simply connected domain in R^n, $n \geq 3$ such that for every cooriented smooth hypersurface the field of Finsler normals is exact. Start with some relevant facts from Finsler geometry – see, e.g., [2].

A Finsler metric is given by a Hamiltonian function $H(q,p)$ in the cotangent bundle (as usual, q are the position and p are the momenta variables); this function is positive, fiber-wise quadratically convex and homogeneous of degree 1 in the p variables. Let S be the unit level hypersurface of H identified with the space of cooriented contact elements in $\mathbf{R}^n$; the contact element, corresponding to $(q,p) \in S$, is the hyperplane Ker p at point q. The space of contact elements carries a canonical contact structure given by the contact form $\lambda = pdq$. The geodesic flow of the Finsler metric is the Hamiltonian vector field u of the function H; it is tangent to S and is the Reeb field: $\lambda(u) = 1$ and $i_u d\lambda = 0$.

A cooriented hypersurface $M \subset \mathbf{R}^n$ lifts to a Legendrian submanifold $L \subset S$ consisting of the contact elements tangent to M. The Finsler normals to M are the projections to $\mathbf{R}^n$ of the vectors of the geodesic flow with the foot points on L.

The next result completes a description started in [5].

Theorem 6.1. *A Finsler metric satisfies the desired property if and only if $H(q,p) = G(p,pq)F(q)$ where G is a positive smooth homogeneous of degree 1 function of $n+1$ variables and F is a positive smooth function of n variables.*

Proof. The Legendrian lift of a hypersurface M consists of the points (q,p) such that $q \in M$, $p(T_qM) = 0$, $H(q,p) = 1$. In particular, p is a conormal field along M. The vector field u being $H_p\partial_q - H_q\partial_p$, the Finsler normals to M are generated by the vectors $v = H_p/H$; due to the Euler equation, one has: $vp = 1$.

The exactness condition for the transverse line field is that the 1-form vdp is exact on M. That is, the form $(\ln H)_p dp$ is exact on L, and therefore the 2-form $\omega = d((\ln H)_p dp) = (\ln H)_{pq} dq \wedge dp$ vanishes on every Legendrian submanifold of S, transverse to the fibers of the projection $S \to \mathbf{R}^n$. It follows that ω belongs to the differential ideal generated by λ. We notice for later use that the condition is conformally invariant: one may multiply H by an arbitrary positive function of q.

Let $\omega = \alpha \wedge \lambda + \phi d\lambda$ where α is a 1-form and ϕ is a function on S. Since ω is closed, one has: $d\alpha \wedge \lambda = (\alpha - d\phi) \wedge d\lambda$. Restricting to the kernel of λ, the left hand side of the above equality vanishes. Since $d\lambda$ is non-degenerate on Ker λ and dim Ker λ is at least 4 one concludes that $\alpha - d\phi = 0$ on Ker λ. Therefore $\alpha = d\phi + \psi\lambda$ where ψ is a function, and it follows that $\omega = d\phi \wedge \lambda + \phi d\lambda = d(\phi\lambda)$.

Since $\omega = -d((\ln H)_q dq)$ and S is simply connected it follows that $(\ln H)_q = -\phi p dq + df$ where f is a function on S. This implies that $f_p = 0$, that is, f depends only on q. Let $\bar{H} = \exp(-f)H$; then $(\ln \bar{H})_q = -\phi p dq$. It follows that $\bar{H}_q = \psi p dq$ where ψ is another function.

Being a homogeneous function, $\bar{H}$ is defined on $T^*\mathbf{R}^n$ with the zero section deleted. Therefore in $T^*\mathbf{R}^n$ one has: $\bar{H}_q = \psi p dq + g d\bar{H}$ where g is a function. Pairing both sides with the Euler field $p\partial p$ yields: $0 = gp\bar{H}_p = g\bar{H}$. Thus $g = 0$ and $\bar{H}_q = \psi p$.

Fix p; then the vector $\bar{H}_q$ is perpendicular to the hyperplanes $pq = const$. Therefore these hyperplanes are the level sets of $\bar{H}$ considered as a function of q. This implies that $\bar{H}(q,p) = G(p, pq)$ where G is a positive smooth homogeneous of degree 1 function of $n+1$ arguments and, finally, $H(q,p) = G(p,pq)F(q)$ with $F = \exp f$.

Conversely, let $H(q,p) = G(p,pq)F(q)$. Denote the derivative of G with respect to the last argument by prime. Then

$$(\ln H)_p dp = d \ln G - (\ln G)_q dq = d \ln G - (\ln G)'(pdq).$$

The form pdq vanishes on every Legendrian manifold L, so the restriction of $(\ln H)_p dp$ to L is exact. This completes the proof. ☐

Example. The hyperbolic metric in the unit ball $q^2 < 1$ whose geodesics are straight lines (Klein model) has the following Hamiltonian:

$$H(q,p) = (1 - q^2)^{1/2}(p^2 - (pq)^2)^{1/2}.$$

This Hamiltonian satisfies the condition of the theorem – compare with property (i), Section 1 .

It would be interesting to give a projective interpretation to the class of Finsler metrics described in Theorem 6.1 An affine characterization of the Hamiltonians $H(q,p) = G(p,pq)$ is that every contact element remains parallel to itself in the geodesic flow.

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