

Geometry of exact transverse line fields and projective billiards

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1. Introduction and survey of results

1.1. Start with a description of the (usual) billiard transformation. A billiard table is a convex domain in \mathbf{R}^n bounded by a smooth closed hypersurface M . The billiard transformation acts on the set of oriented lines (rays) in \mathbf{R}^n that intersect M , and is determined by the familiar law of geometric optics: the incoming and the outgoing rays lie in one 2-plane with the normal line to M at the impact point and make equal angles with this normal line. Equivalently, the reflection can be described as follows: at the impact point, the tangential component of a velocity vector along a ray is preserved while the normal component changes sign. An analogous law describes the billiard reflection in a Riemannian manifold with the boundary M . An interested reader may find a survey of the theory of mathematical billiards in [T 1].

The billiard transformation is defined in metric terms and is equivariant with respect to the isometries of \mathbf{R}^n that naturally act on the space of rays (and transform billiard tables to isometric ones). In [T 2] we introduced a more general dynamical system, the projective billiard, equivariant under the greater group of projective transformations of \mathbf{R}^n . To define the projective billiard transformation one needs an additional structure: a smooth transverse line field ξ along the billiard hypersurface $M \subset \mathbf{R}^n$ (respectively, a diffeomorphism f of \mathbf{R}^n transforms a pair (M, ξ) to $(f(M), Df(\xi))$ where Df is the derivative of f).

Given such a "hairy" billiard table, the law of the projective billiard reflection reads:

- 1). *The incoming ray, the outgoing ray and the transverse line $\xi(x)$ at the impact point x lie in one 2-plane π ;*
- 2). *The three above lines and the line of intersection of π with the tangent hyperplane $T_x M$ constitute a harmonic quadruple of lines.*

(Four lines through a point is a harmonic quadruple if the cross-ratio of their intersection points with some, and then every, auxiliary line equals -1). Equivalently, a velocity

vector along the incoming ray is decomposed at the impact point x into the tangential and transverse components, the latter being collinear with $\xi(x)$; then the transverse component changes sign yielding a velocity vector along the outgoing ray.

If the transverse line field consists of the normals to M then the projective billiard coincides with the usual one. More generally, consider a metric g in a domain containing M whose geodesics are straight lines. Then the (usual) billiard in this metric is a projective billiard with the transverse line field consisting of the g -normals to M . Examples of such metrics are the spherical and the hyperbolic ones; conversely, a metric in a domain in \mathbf{R}^n whose geodesics are straight lines is a metric of constant curvature (see [D]). To further generalize, one may consider a Finsler metric whose geodesics are straight lines. Given a hypersurface M , the billiard transformation of the set of rays intersecting M is defined in this setting as well but, unless the Finsler metric is a Riemannian one, it is not a projective billiard transformation.

The space of rays in \mathbf{R}^n has a canonical symplectic structure obtained from the one in the cotangent bundle $T^*\mathbf{R}^n$ by symplectic reduction (see, e.g., [A-G]). A fundamental property of the billiard transformation is that it preserves this symplectic form. Turning from the Hamiltonian to the Lagrangian point of view, the billiard trajectories are the extremals of the length functional. These properties make it possible to use methods of symplectic geometry and classical mechanics in the study of the billiard dynamics.

It is therefore natural to ask for which pairs (M, ξ) the phase space of the respective projective billiard transformation T has a T -invariant symplectic structure. This difficult problem is far from being solved; some results in this direction from [T 2,3] are surveyed in the next section. These results suggest that the following class of transverse fields plays a special role in the projective billiard problem, similar to that of the normal fields in the usual billiard setting.

The following notation will be used throughout the paper. Let $n(x)$, $x \in M$ denote the unit normal vector field along a cooriented hypersurface $M \subset \mathbf{R}^n$. Given a transverse line field ξ along M , let v be the vector field along ξ such that the scalar product $v(x) \cdot n(x) = 1$ for all $x \in M$.

Definition. A transverse field ξ is called *exact* if the 1-form $v \cdot n$ on M is exact.

Here and elsewhere we adopt the following convention: v and n are thought of as vector-valued functions, and $v \cdot n$ is the scalar product of a vector-function and a vector-valued 1-form.

Note that the form $v \cdot n$ does not depend on the sign of the vector n . Thus the definition of exactness makes sense for non-cooriented hypersurfaces as well. It also extends to the

case when M is an immersed hypersurface and, more generally, a wave front. A wave front is the projection to \mathbf{R}^n of a smooth immersed Legendrian submanifold L^{n-1} of the space of contact elements in \mathbf{R}^n ; "Legendrian" means that L is tangent to the canonical contact structure in the space of contact elements (see [A-G]). A smooth hypersurface M canonically lifts to such a Legendrian manifold by assigning the tangent hyperplane $T_x M$ to each point $x \in M$. Although a wave front may have singularities, it has a well defined tangent hyperplane at every point, so, up to a sign, the unit normal vector is defined too. Then vdn is considered as a 1-form on L , and the field v along M is called exact if this 1-form on L is exact.

Geometry of exact transverse line fields is closely related to symplectic geometry of systems of rays, Finsler geometry and projective geometry. We find it of interest on its own.

1.2. In this section we survey some results obtained in [T 2-5]. Start with properties of exact transverse fields. Although they are defined in Euclidean terms, exactness is a projective property (see [T 3]).

Theorem 1.2.1. *Let ξ be an exact transverse line field along $M \subset \mathbf{R}^n$, and f be a projective transformation of \mathbf{R}^n whose domain contains M . Then the line field $Df(\xi)$ along $f(M)$ is also exact.*

This result is sharp: we will show in Section 2 that if a local diffeomorphism f of \mathbf{R}^n takes every exact transverse field along every hypersurface to an exact field then f is projective.

Recall that a (not necessarily symmetric) Finsler metric in \mathbf{R}^n is given by a smooth field of smooth star-shaped quadratically convex hypersurfaces $S_x \subset T_x \mathbf{R}^n$, $x \in \mathbf{R}^n$. These hypersurfaces, called the indicatrices, consist of the Finsler unit vectors and play the role of the unit spheres in Riemannian geometry. A Minkowski metric in \mathbf{R}^n is a translation-invariant Finsler metric. Given a cooriented hypersurface $M \subset \mathbf{R}^n$, the Finsler normal to M at point $x \in M$ is defined as follows. There is a unique $u \in S_x$ such that the outward cooriented tangent hyperplane $T_u(S_x)$ is parallel to and has the same coorientation as $T_x M$ in $T_x \mathbf{R}^n$. By definition, this u is the Finsler normal to M at x .

It is not true, in general, that the Finsler normals to a hypersurface constitute an exact transverse field. However, the next result holds (see [T 3]).

Theorem 1.2.2. *Given a smooth (not necessarily symmetric) Minkowski metric in \mathbf{R}^n and a cooriented hypersurface $M \subset \mathbf{R}^n$, the field of Minkowski normals to M is exact.*

The next property of exact transverse fields, implicit in [T 3], generalizes the Huygens

principle of wave propagation (and coincides with it when the exact field consists of the normals).

Theorem 1.2.3. *Let v be the vector field along a hypersurface M generating an exact transverse line field such that $vn = 1$, and let f be a function on M such that $vdn|_M = df$. Fix a real t . For $x \in M$, let $x_t = x + te^{f(x)}v(x)$, and let M_t be the locus of points x_t . Then the tangent hyperplane to M_t at its every smooth point x_t is parallel to the tangent hyperplane to M at x .*

In the case of the usual wave propagation, that is, when $v = n$, the hypersurface M_t is t -equidistant from M . This M_t is a wave front, and the corresponding Legendrian manifold L_t is imbedded and consists of the tangent hyperplanes T_xM , parallel translated to the respective points x_t . A similar description applies to an exact transverse line field; however, the manifold L_t may be singular in the general case.

In the next two theorems $M \subset \mathbf{R}^2$ is a closed strictly convex smooth plane curve. The following result is contained in [T 2, 4].

Theorem 1.2.4. *A transverse line field ξ along M is exact if and only if there exists a parameterization $M(t)$ such that $\xi(M(t))$ is generated by the acceleration vector $M''(t)$ for all t .*

Another specifically 2-dimensional result from [T 4, 5] is a generalization of the classical 4-vertex theorem to exact transverse fields along curves (the 4-vertex theorem is the case of the normals).

Theorem 1.2.5. *Let ξ be a generic exact transverse line field along a closed convex curve M . Then the envelope of the 1-parameter family of lines $\xi(x)$, $x \in M$ has at least 4 cusp singularities.*

Next we turn to the projective billiards. We already mentioned a class of projective billiard transformations with an invariant symplectic form, the usual billiards in a domain in \mathbf{R}^n with a metric g of constant curvature whose geodesics are straight lines. The transverse field is the field of g -normals, and the invariant symplectic form is the canonical symplectic form on the space of oriented geodesics. Applying a projective transformation to such a billiard one again obtains a projective billiard with an invariant symplectic structure.

A different class of projective billiards with an invariant symplectic structure was introduced in [T 3]. Let $M \subset \mathbf{R}^n$ be a sphere equipped with an exact transverse line field. Identify the interior of M with the hyperbolic space \mathbf{H}^n so that the chords of M represent

the straight lines in \mathbf{H}^n (the Beltrami-Klein model of hyperbolic geometry). Then the respective projective billiard transformation T acts on oriented lines in \mathbf{H}^n , and the phase space of T is not compact, unlike that of the usual billiard transformation.

Theorem 1.2.6. *The canonical symplectic structure on the space of rays in \mathbf{H}^n is T -invariant. The map T has n -periodic orbits for every $n \geq 2$.*

Note that the supply of these projective billiards inside a sphere is, roughly, the same as that of the usual billiards: modulo a finite dimensional correction, both depend on a function on the sphere S^{n-1} (for the usual billiards, this is the support function of the billiard hypersurface; for the projective ones, it is the function f such that $v dn = df$).

This projective billiard inside a sphere can be described as follows. A transverse field along M can be thought of as a smooth family of rays L in \mathbf{H}^n such that for every point x at infinity there is a unique ray $l_x \in L$ whose forward limit is x . Let σ_x be the involution of the space of rays in \mathbf{H}^n that reflects the rays in l_x , and let τ be the involution of this space that reverses the orientation of the rays. Then, for a ray r with the forward limit point x , one has: $T(r) = \tau(\sigma_x(r))$.

The next example is discussed in [T 2]. Let $M \subset \mathbf{R}^2$ be a smooth closed convex curve and ξ be the field of lines along M through a fixed point x inside M . Then the corresponding projective billiard transformation has an invariant area form which blows up on the curve in the phase space that consists of the lines through x (see [T 2] for details and, in particular, the relation with the dual, or outer, billiards). This construction generalizes to higher dimensions but the dynamics is rather dull: every orbit stays in a 2-plane.

Finally, the next result from [T 2] is another manifestation of the relation between projective billiards and exact transverse line fields. Let M be smooth closed convex plane curve with a transverse line field ξ . The phase space of the projective billiard map T is a cylinder whose two boundary circles consist of the oriented lines tangent to M .

Theorem 1.2.7. *If there exists (the infinite jet of) a T -invariant area form along a boundary component of the phase cylinder then ξ is exact.*

1.3. In this section we outline the contents of the present paper.

Section 2 concerns properties of exact transverse line fields. First, we prove a result, partially converse to Theorem 1.2.2. Let $v(x)$, $x \in M$ be an exact transverse field along a quadratically convex closed hypersurface M , and assume that the Gauss map

$$G : M \rightarrow S^{n-1}, G(x) = v(x)/|v(x)|$$

is a diffeomorphism. Then there exists a Minkowski metric in \mathbf{R}^n (in general, not symmetric) such that the transverse field is the field of Minkowski normals to M (the particular case of curves in the plane was considered in [T 3]).

Next, we discuss the following local question: for which Finsler metrics in \mathbf{R}^n the field of the Finsler normals to every hypersurface is exact? We give a certain technical criterion for this to hold and deduce from it that the field of normals to a hypersurface in the hyperbolic or spherical metric in \mathbf{R}^n , whose geodesics are straight lines is exact.

We also show that the field of the affine normals to a convex hypersurface is exact (the particular case of plane curves was considered in [T 3]). Given a convex hypersurface $M \subset \mathbf{R}^n$ and a point $x \in M$, consider the $(n - 1)$ -dimensional sections of M by the hyperplanes, parallel to $T_x M$. The centroids of these sections lie on a curve starting at x , and the tangent line to this curve at x is, by definition, the affine normal to M at x .

We expect the exact transverse fields to enjoy many properties of the Euclidean normals. In particular, given an immersed closed hypersurface $M \subset \mathbf{R}^n$ and a point $p \in \mathbf{R}^n$, the number of normals to M from p is bounded below by the least number of critical points of a smooth function on M , and, generically, by the sum of the Betti numbers of M (the function in question is, of course, the distance squared from p to M). We extend this result to an exact transverse line field ξ along an immersed locally quadratically convex closed hypersurface M : for every point p the number of lines $\xi(x)$, $x \in M$ through p has the same lower bound as in the case of normals.

The nondegeneracy condition on the second quadratic form of M is used in the proof but may be redundant. We make a bolder conjecture. Let M be an immersed closed hypersurface in \mathbf{R}^n and L the corresponding Legendrian submanifold in the space of oriented contact elements in \mathbf{R}^n . Let L_1 be a Legendrian manifold, Legendrian isotopic to L , and M_1 its wave front. *Conjecture*: for every exact transverse line field ξ along M_1 and a point $p \in \mathbf{R}^n$ the number of lines $\xi(x)$, $x \in M$ through p is bounded below by the least number of critical points of a smooth function on M . This estimate holds for the Euclidean normals and, more generally, the normals in a Finsler Hadamard metric – see [Fe 1].

One may ask a similar question concerning the least number of "diameters" of a hypersurface M equipped with an exact transverse field ξ . A diameter of (M, ξ) is a line l , intersecting M at points x, y , such that $\xi(x) = \xi(y) = l$. We failed to extend the results from [Fe 2, Pu] on the Euclidean diameters (double normals) to exact transverse fields.

Next, we turn to the projective invariance of exactness. The proof of Theorem 1.2.1 given in [T 3] is purely computational. We give an intrinsic definition of exact transverse line fields along hypersurfaces in the projective space. This definition agrees with the

previous one for hypersurfaces that lie in an affine chart, and this implies Theorem 1.2.1. We extend the lower bound for the number of lines from an exact transverse field along a closed hypersurface through a fixed point to hypersurfaces in \mathbf{RP}^n which are quadratically nondegenerate. An example of such a surface is the hyperboloid in \mathbf{RP}^3 or its small perturbation.

Finally, we consider a smooth transverse line field ξ along a sphere. Identifying the interior of the sphere with \mathbf{H}^n , one has an imbedding of the sphere to the space of oriented lines in \mathbf{H}^n : to a point $x \in S^{n-1}$ there corresponds the line $\xi(x)$, oriented outward. Let L be the image of this map. We show that if $n \geq 3$ then ξ is exact if and only if L is a Lagrangian submanifold of the space of oriented lines in \mathbf{H}^n with its canonical symplectic structure, associated with the hyperbolic metric; if $n = 2$ then the condition is that L is exact Lagrangian for an appropriate choice of the 1-form whose differential is the symplectic form. It follows that if ξ is exact then the lines from this field are the hyperbolic normals to a one-parameter family of equidistant closed wave fronts in \mathbf{H}^n . The case of an exact line field ξ along a circle in the plane is of interest. The envelope of the lines from ξ is then the caustic of a closed curve, and the algebraic length of this envelope in the hyperbolic metric equals zero (the sign of the length changes after each cusp).

The above observation offers a new look on the projective billiards inside a sphere, associated with exact transverse line fields. Let $M \subset \mathbf{H}^n$ be a closed convex hypersurface, ξ the family of its outward hyperbolic normals and M_t its t -equidistant hypersurface. Then ξ determines an exact transverse field along the sphere at infinity, and the respective projective billiard inside this sphere may be considered, rather informally, as the limit $t \rightarrow \infty$ case of the usual billiard inside M_t .

Section 3 concerns projective billiards. Our first observation is as follows. Consider a closed quadratically convex hypersurface M with an exact transverse line field ξ , and let A and B be points inside M . Then there exist at least two rays through A that, after the projective billiard reflection in M , pass through B (the case of plane curves was considered in [T 2]). If the points A and B coincide one obtains two lines from ξ passing through a given point whose existence was already mentioned above.

The hyperbolic distance between points A and B inside a ball in the Beltrami-Klein model is $|\ln[A, B, X, Y]|$ where $[\]$ denotes the cross-ratio, and X, Y are the points of intersection of the line AB with the boundary sphere. The same formula defines a Finsler metric inside any convex closed hypersurface M . This metric is called the Hilbert metric, and its geodesics are straight lines. The Hilbert metric determines a canonical symplectic structure ω on the space of oriented lines that intersect M . If M is a sphere with an exact

transverse field ξ then, by Theorem 1.2.6, the respective projective billiard transformation preserves the form ω . It is an interesting problem to describe all pairs (M, ξ) such that the corresponding projective billiard map preserves this symplectic structure. We prove this to be the case for a closed convex plane curve M if and only if M is an ellipse and ξ is an arbitrary transverse field.

The phase space of the projective billiard inside a circle is a cylinder. This cylinder, being the space of rays in \mathbf{H}^2 , has a canonical area form, preserved by the projective billiard transformation for every transverse line field along the circle. The flux of an area-preserving transformation T of a cylinder C is defined as follows. Choose a noncontractible simple closed curve $\gamma \subset C$. Then the flux of T is the signed area between γ and $T(\gamma)$, and this area does not depend on the choice of γ . We show that the projective billiard transformation in a circle, associated with a transverse field ξ , has zero flux if and only if ξ is exact. It follows that if this transformation has an invariant circle then the corresponding transverse field is exact. We construct an example to show that the last claim may fail for plane curves, other than circles. We also interpret the projective billiard inside a circle as a dynamical system on the one-sheeted hyperboloid.

The mirror equation of geometric optics describes the following situation. Suppose that an infinitesimal beam of rays from a point A reflects at a point X of a smooth billiard curve M and focuses at a point B . Let a and b be the distances from A and B to X , let k be the curvature of M at X and α the angle of incidence of the ray AX . The mirror equation reads:

$$\frac{1}{a} + \frac{1}{b} = \frac{2k}{\sin \alpha}.$$

This equation proved very useful in the study of billiards – see [W 1, 2].

We find an analog of the mirror equation for the projective billiard in the unit circle M . Parameterize M by the usual angle parameter t . Denote by $\phi(t)$ the angle made by the line of the transverse field at point $M(t)$ with the tangent vector $M'(t)$, and let ϕ and ϕ' be its value and the value of its derivative at X . Let a and b have the same meaning as before, and α and β be the angles made by the rays XA and XB with the positive direction of M at X . Then

$$\frac{2\phi'}{\sin^2 \phi} + \frac{1}{\sin^2 \alpha} + \frac{1}{\sin^2 \beta} = \frac{1}{a \sin \alpha} + \frac{1}{b \sin \beta}.$$

A similar equation can be written for every plane projective billiard.

At the end of Section 3 we present a conjecture describing all the cases in which a plane projective billiard is integrable.

2. Exact transverse line fields

2.1. We use the same notation as in the Introduction. Let $M \subset \mathbf{R}^n$ be a smooth cooriented hypersurface and v a vector field along M that defines an exact transverse line field. One has: $vn = 1$ and $vdn|_M = df$. Consider the smooth map $g : M \rightarrow \mathbf{R}^n$ given by the formula: $g(x) = e^{f(x)}v(x)$. Let $N = g(M)$.

Lemma 2.1.1. *At every smooth point of N the tangent hyperplane $T_{g(x)}N$ is parallel to T_xM .*

Proof. One wants to show that $n(x)$ is orthogonal to $T_{g(x)}N$, that is, the 1-form $n(x) dg(x)$ vanishes on N . Since $vn = 1$ one has: $vdn + ndv = 0$. Therefore

$$ndg = e^{f(x)}n(vdf + dv) = e^{f(x)}(df + ndv) = e^{f(x)}(df - vdn) = 0,$$

and we are done.

Geometrically this lemma means the following. Consider a homogeneous codimension one distribution in the punctured \mathbf{R}^n whose hyperplanes along the ray, spanned by $v(x)$, are parallel to T_xM . Then this distribution is integrable, the leaves being homothetic to the hypersurface N .

In general, N may be singular. Assume, however, that M is a closed quadratically convex hypersurface and the Gauss map

$$G : M \rightarrow S^{n-1}, G(x) = v(x)/|v(x)|$$

is a diffeomorphism.

Theorem 2.1.2. *There exists a (not necessarily symmetric) Minkowski metric in \mathbf{R}^n such that the exact transverse field v is the field of Minkowski normals to M .*

Proof. Our assumptions imply that N is a quadratically convex star-shaped closed hypersurface. The desired Minkowski metric has N as its indicatrix.

The support function of a convex star-shaped hypersurface in \mathbf{R}^n is the distance from the origin to its tangent hyperplanes. The value of the support function of the indicatrix N at point $g(x) \in N$ equals

$$n(x) \cdot e^{f(x)}v(x) = e^{f(x)}.$$

In general, the construction of this section gives a smooth map from M to the space of contact elements in \mathbf{R}^n : to $x \in M$ there corresponds the hyperplane T_xM based at the point $e^{f(x)}v(x)$. The image of this map is Legendrian, but it may be singular.

2.2. Using the same notation as before, call a transverse line field along a hypersurface *closed* if the 1-form $v dn$ is closed on M . Unlike exactness, this is a local property: a field is closed iff so is its restriction to every open subset of M . For simply connected hypersurfaces the two notions are equivalent.

In this section we discuss the Finsler metrics in domains in \mathbf{R}^n such that the field of the Finsler normals to every smooth hypersurface is closed.

A Finsler metric can be characterized by a positive function $H(q, p)$ in $T^*\mathbf{R}^n$, strictly convex and homogeneous of degree 2 in the momentum variable p – see [Ar]. The unit level hypersurface of H in each fiber of the cotangent bundle consists of the Finsler unit covectors, and the Hamiltonian vector field $sgrad H$ is the geodesic flow in the Finsler metric. Given a point q of a cooriented hypersurface $M \subset \mathbf{R}^n$, let $p \in T_q^*\mathbf{R}^n$ be a covector such that $Ker p = T_q M$ and p gives $T_q M$ the positive coorientation. The Finsler normal $\xi(q)$ to M at q is spanned by the projection to \mathbf{R}^n of the vector $sgrad H(q, p)$; since H is homogeneous the line $\xi(q)$ does not depend on the choice of p .

Theorem 2.2.1. *The field of the Finsler normals ξ along every hypersurface $M \subset \mathbf{R}^n$ is closed if and only if the 2-form $\Omega = (\ln H)_{pq} dp \wedge dq$ belongs to the ideal $(pdp, pdq, dp \wedge dq)$ in the algebra of differential forms in $T^*\mathbf{R}^n$.*

Here we use the obvious notation:

$$F_{pq} dp \wedge dq = \sum \frac{\partial^2 F}{\partial p_i \partial q_j} dp_i \wedge dq_j, \quad dp \wedge dq = \sum dp_i \wedge dq_i,$$

etc.

Proof. Identify the tangent and cotangent spaces by the Euclidean structure. Then the unit normal vector $n(q)$, $q \in M$ is considered as a covector p such that $Ker p = T_q M$. The projection of $sgrad H$ to \mathbf{R}^n is the vector H_p . Hence the field ξ is generated by the vectors $H_p(q, p)$, $q \in M$, $p = n(q)$. Let $v(q) = H_p/2H$; by the Euler formula, $vn = pH_p/2H = 1$.

Thus one wants the 2-form

$$d(vdn) = d((\ln H)_p dp)/2 = (\ln H)_{pq} dq \wedge dp/2$$

to vanish on every hypersurface M whenever p is the unit normal vector field along M .

Assume that $\Omega \in (pdp, pdq, dp \wedge dq)$. If p is the unit normal field along M then $p^2 = 1$ and $pdq|_M = 0$ for $q \in M$. It follows that $pdp|_M = 0$ and $dp \wedge dq|_M = 0$; hence $\Omega|_M = 0$.

Conversely, if $L^{n-1} \subset T^*\mathbf{R}^n$ is a submanifold such that $pdp|_L = 0$, $pdq|_L = 0$ and L projects diffeomorphically to a hypersurface $M \subset \mathbf{R}^n$ then L consists of the pairs (q, p)

where $q \in M$ and p is a normal vector field to M of constant length. If $\Omega|_L = 0$ for every such L then Ω belongs to the differential ideal generated by pdp and pdq , that is, $\Omega \in (pdp, pdq, dp \wedge dq)$.

Notice that the condition of Theorem 2.2.1 is conformally invariant: if a Hamiltonian function $H(q, p)$ satisfies this condition then so does $f(q)H(q, p)$ for every positive function $f(q)$.

Corollary 2.2.2. *Let g be a metric of constant curvature in a domain in \mathbf{R}^n whose geodesics are straight lines. Then the field of g -normals to every hypersurface is exact.*

Proof. Consider the case of the hyperbolic metric, the spherical case being analogous.

Recall the construction of the hyperbolic metric in the unit ball. Let H be the upper sheet of the hyperboloid $x^2 - y^2 = -1$ in $\mathbf{R}_x^n \times \mathbf{R}_y^1$ with the Lorentz metric $dx^2 - dy^2$. The restriction of the Lorentz metric to H is a metric of constant negative curvature. Project H from the origin to the hyperplane $y = 1$. Let q be the Euclidean coordinate in this hyperplane. The projection is given by the formula:

$$x = \frac{q}{(1 - q^2)^{1/2}}, \quad y = \frac{1}{(1 - q^2)^{1/2}},$$

and the image of H is the open unit ball $q^2 < 1$. The hyperbolic metric g in the unit ball is given by the formula:

$$g(u, v) = \frac{uv}{1 - q^2} + \frac{(uq)(vq)}{(1 - q^2)^2},$$

where u and v are tangent vectors at q .

The Hamiltonian function $H(q, p)$ is the corresponding metric on the cotangent space. Lifting the indices yields the following formula:

$$H(q, p) = (1 - q^2)(p^2 - (pq)^2).$$

Therefore

$$vdn = \frac{1}{2}(\ln H)_p dp = \frac{pdp - (pq)(qdp)}{2(p^2 - (pq)^2)} = \frac{1}{4}d(\ln(p^2 - (pq)^2))$$

which is an exact 1-form.

2.3. This section concerns the number of lines from an exact transverse field passing through a fixed point in \mathbf{R}^n . Let $M \subset \mathbf{R}^n$ be a closed locally quadratically convex immersed hypersurface and ξ an exact transverse line field along M . Fix a point $p \in \mathbf{R}^n$.

Theorem 2.3.1. *The number of points $x \in M$ such that the line $\xi(x)$ passes through p is greater than or equal to the least number of critical points of a smooth function on M .*

Proof. Without loss of generality, assume that p is the origin. Due to the local convexity, M is coorientable. As before, let v be the vector field along ξ such that $vn = 1$, $vdn = df$. Consider the function on M given by the formula: $h(x) = e^{-f(x)}(x \cdot n(x))$.

We claim that if x is a critical point of this function then $\xi(x)$ contains the origin. One has:

$$e^f dh = d(xn) - (xn)df = xdn - (xn)(vdn) = (x - (xn)v)dn;$$

the second equality is due to the fact that $ndx = 0$ on M .

Suppose that $dh(x) = 0$; then $(x - (xn)v)dn = 0$. Notice that the vector $x - (xn)v$ is orthogonal to $n(x)$, thus $x - (xn)v \in T_x M$. The value of the second quadratic form of M on two tangent vectors $u, w \in T_x M$ is $(udn(x))(w)$. Since M is quadratically nondegenerate, $udn = 0$ on $T_x M$ only if $u = 0$. Applying this to $u = x - (xn)v$ one concludes that $x = (x \cdot n(x))v(x)$. It follows that the line $\xi(x)$, spanned by $v(x)$, contains the origin.

The next construction associates with a point p a hypersurface N such that the lines $\xi(x)$, $x \in M$, passing through p , correspond to the perpendiculars from p to N . Consider the smooth map $F : M \rightarrow \mathbf{R}^n$ given by the formula:

$$F(x) = e^{-f(x)}[x + (x \cdot n(x))(n(x) - v(x))],$$

and let N be its image.

Lemma 2.3.2. a). *The tangent hyperplane to N at its smooth point $F(x)$ is parallel to $T_x M$.*

b). *A point $p \in \mathbf{R}^n$ lies on the line $\xi(x)$ if and only if p lies on the normal line to N at point $F(x)$ (if $F(x)$ is not a smooth point of N we mean the line, orthogonal to the hyperplane, parallel to $T_x M$ and based at $F(x)$).*

c). *The value of the support function of N at point $F(x)$ is $e^{-f(x)}(x \cdot n(x))$ (with the same convention if $F(x)$ is not a smooth point of N).*

Proof. To prove a) one needs to show that the 1-form $n(x)dF(x)$ vanishes on M . One has:

$$e^f dF = dx + (xn)(dn - dv) + (n - v)(ndx + xdn) - [x + (xn)(n - v)](vdn).$$

It follows that

$$e^f ndF = ndx + (xn)(ndn - ndv) + (n \cdot (n - v))(ndx + xdn) - (n \cdot [x + (xn)(n - v)])(vdn).$$

Since $ndx = 0$, $ndn = 0$, and $(n \cdot (n - v)) = 0$ one concludes:

$$e^f ndF = -(xn)(ndv) - (nx)(vdn) = -(xn)d(nv) = 0.$$

As before, let p be the origin. If $p \in \xi(x)$ then $x = cv(x)$ for some constant c . Scalar multiply by $n(x)$ to find that $c = x \cdot n(x)$. Thus $x = (xn)v$, and

$$F(x) = e^{-f(x)}[x + (xn)(n - v)] = e^{-f(x)}(xn)n(x).$$

It follows that the vector $F(x)$ is collinear with $n(x)$.

Conversely, if $F(x)$ is collinear with $n(x)$ then $x + (xn)(n - v) = cn$ for some constant c . One again finds that $c = x \cdot n(x)$. Thus $x + (xn)(n - v) = (xn)n$, and $x = (xn)v(x)$. Therefore the vector x is collinear with $v(x)$, and the claim b) follows.

Finally the support function of N is $n(x) \cdot F(x) = e^{-f(x)}(x \cdot n(x))$. The lemma is proved.

Similarly to Section 2.1, the manifold N is, in general, singular. One may construct a Legendrian submanifold in the space of contact elements in \mathbf{R}^n whose projection to \mathbf{R}^n is N but this Legendrian manifold may have singularities as well.

2.4. In this section we discuss intrinsic definitions of exact transverse line fields along hypersurfaces in affine and projective spaces.

Recall that the definition of the conormal bundle of a hypersurface M in an affine space V :

$$\nu(M) = \{(q, p) \in T^*V \mid q \in M, \text{Ker } p = T_q M\}.$$

A section of $\nu(M)$ will be called a conormal field along M . A conormal field is considered as a covector-valued function $M \rightarrow V^*$.

Lemma 2.4.1. *A transverse line field ξ along a coorientable hypersurface M is exact if and only if there exists a conormal field p along M such that for some, and then every, vector field u generating ξ one has: $udp = 0$.*

Here udp is the 1-form on M resulting in pairing of a vector-valued function and a covector-valued 1-form.

Proof. Introduce a Euclidean structure in V and identify V with V^* . Let ξ be exact, that is, ξ is generated by a vector field v such that $vn = 1$ and $vdn = df$. Consider the covector field $p = e^{-f}n$. One has:

$$vdp = e^{-f}v(dn - ndf) = e^{-f}(vdn - df) = 0.$$

Conversely, if p is a conormal field then $p = \phi n$ for some function ϕ . Given a vector field u along M such that $udp = 0$, assume, without loss of generality, that $up = 1$. Let $v = \phi u$. Then $vn = 1$ and

$$vdn = vd(\phi^{-1}p) = v(\phi^{-1}dp - \phi^{-2}pd\phi) = -\phi^{-1}d\phi = -d \ln \phi.$$

Thus the transverse line field generated by u is exact.

Recall the notion of polar duality. Let S be a star-shaped hypersurface in a linear space V . To a vector $x \in S$ there corresponds the covector $p_x \in V^*$ such that $xp_x = 1$ and $\text{Ker } p_x = T_x S$. The locus of p_x , $x \in S$ is a hypersurface $S^* \subset V^*$, polar dual to S . By construction, $p_x dx = 0$ on S .

The relation of exactness with polar duality is as follows. Let $p : M \rightarrow V^*$ be a conormal field along a hypersurface $M \subset V$. Generically, $p(M)$ is star-shaped, and its polar dual hypersurface is given by the map $u : M \rightarrow V$ where $up = 1$ and $udp = 0$. Thus u , considered as a vector field along M , generates an exact transverse line field. Polar duality also explains the construction of the Minkowski metric in Section 2.1: the indicatrix N is polar dual to the hypersurface $p(M)$ where $p = e^{-f}n$ is the conormal field corresponding to the given exact transverse field along M . The hypersurface $p(M)$ is the unit level surface of the Hamiltonian defying the Minkowski metric (called the figuratrix).

Apply Lemma 2.4.1 to the affine normals of a convex hypersurface in an affine space defined in Section 1.3.

Corollary 2.4.2. *The field of affine normals is exact.*

Proof. Identify vectors and covectors by a Euclidean structure. Let n be the unit normal field to M^m and K its Gauss curvature. Consider the conormal field $p = K^{-1/(m+2)}n$, and let u be a vector field generating the affine normals to M . Then, according to [Ca], $udp = 0$, and the result follows from Lemma 2.4.1.

Consider the projective space $P(V)$, and let $\pi : V - \{O\} \rightarrow P(V)$ be the projection. Denote by E the Euler vector field in V ; this field generates the fibers of π . Considered as a map $V \rightarrow V$, the field E is the identity.

Let N be a hypersurface in $P(V)$ and $M = \pi^{-1}(N) \subset V$. Given a transverse line field ξ along N , consider a line field η along M that projects to ξ .

Definition. A field ξ is called exact if there exists a homogeneous of degree zero conormal field p along M such that $vdp = 0$ for every vector field v generating η .

The next lemma justifies this definition.

Lemma 2.4.3. a). *The definition is correct, that is, does not depend on the choice of η .*
b). *For a transverse line field along a hypersurface in an affine chart of $P(V)$ the affine and projective definitions of exactness are equivalent.*

Proof. Let v be a vector field along M generating η , and p be a conormal field such that $vd p = 0$ on M . Any other lift η_1 is generated by the vector field $v_1 = v + \phi E$ where ϕ is a function. Notice that $Ep = 0$ since E is tangent to M . It follows that $Edp + pdE = 0$. Since p is conormal to M and $E(x) = x$ one has: $pd x = pdE = 0$ on M . Thus $Edp = 0$. It follows that $v_1 d p = 0$, and η_1 is exact. The claim a) is proved.

To prove b) chose coordinates (x_0, x_1, \dots, x_n) in V and identify the affine part of $P(V)$ with the hyperplane $x_0 = 1$. Decompose V into $\mathbf{R}_{x_0}^1 \oplus \mathbf{R}_x^n$, $x = (x_1, \dots, x_n)$; the vectors and covectors in V will be decomposed accordingly.

Assume that N belongs to the hyperplane $x_0 = 1$. Let u be a vector field along N generating a transverse line field, exact in the affine sense, and p be a conormal field along N such that $ud p = 0$. Consider the covector field

$$P(x_0, x) = \left(-\frac{x}{x_0} \cdot p\left(\frac{x}{x_0}\right), p\left(\frac{x}{x_0}\right) \right)$$

along $M = \pi^{-1}(N)$. Then $E(x_0, x)P(x_0, x) = 0$, so P is a conormal field along M . Lift u to the horizontal vector field $v(x_0, x) = (0, u(x/x_0))$. Then $vdP = u(x/x_0)dp(x/x_0) = 0$. Thus the transverse field generated by u is exact in the projective sense.

Conversely, assume that $v(x_0, x) = (0, u(x/x_0))$ generates a line field along M , exact in the projective sense. The respective conormal field along M is homogeneous of degree zero. Therefore its \mathbf{R}_x^n -component $p(x/x_0)$ defines a conormal field along N , and $ud p = 0$ on N . Thus u generates a line field, exact in the affine sense.

Note that the above lemma implies Theorem 1.2.1, the projective invariance of exactness. We add to it the following remark.

Lemma 2.4.4. *If a local diffeomorphism f of \mathbf{R}^n takes every exact transverse field along every hypersurface to an exact field then f is a projective transformation.*

Proof. The unit normal vector field along a hyperplane is constant, so every transverse line field along it is exact. Therefore f takes hyperplanes to hyperplanes, and it follows that f is projective.

We now extend Theorem 2.3.1 to quadratically nondegenerate hypersurfaces in the projective space.

Theorem 2.4.5. *Let N be a closed cooriented quadratically nondegenerate immersed hypersurface in the projective space $P(V)$ and ξ an exact transverse line field along N .*

Then for every point $a \in P(V)$ the number of points $x \in N$ such that the line $\xi(x)$ passes through a is greater than or equal to the least number of critical points of a smooth function on N .

Proof. We will repeat the argument from the proof of Theorem 2.3.1. Introduce a Euclidean structure in V . Let η be a lift of ξ to $M = \pi^{-1}(N)$ and v a vector field along M normalized so that $vn = 1$ where n is the unit normal field along M . Let p be the homogeneous of degree zero conormal field along M such that $vd p = 0$. Then, by Lemma 2.4.1, $p = e^{-f}n$ where f is a homogeneous of degree zero function on M , and $vd n = df$. Choose a point $A \in V$ that projects to $a \in P(V)$.

Consider the function $h(x) = e^{-f(x)}(A - x) \cdot n(x)$ on M . The same argument as in Theorem 2.3.1 shows that if $dh = 0$ then the vector $(A - x) - ((A - x) \cdot n)v(x)$ is in the kernel of the second quadratic form of M at x . This kernel is generated by the vector $E(x) = x$. Thus $A - x$ belongs to the span of the vectors $v(x)$ and $E(x)$. Projecting back to $P(V)$ one concludes that the line $\xi(\pi(x))$ passes through the point a .

Since M is conical, $xn(x) = 0$; thus $h(x) = e^{-f(x)}A \cdot n(x)$ is homogeneous of degree zero. Therefore this function descends to a function on N , and the result follows.

2.5. This section concerns exact transverse line fields along the unit sphere S^{n-1} .

We start with computing the canonical symplectic structure ω on the space of rays intersecting a closed convex hypersurface $M \subset \mathbf{R}^n$, associated with the Hilbert metric inside M . If $n > 2$ the space of rays is simply connected, and the cohomology class of a 1-form λ such that $d\lambda = \omega$ is uniquely defined. If $n = 2$ the space of rays is a cylinder, and we choose λ so that the curve consisting of the rays through a fixed point inside M is an exact Lagrangian manifold, which means that the restriction of λ on this manifold is exact. An oriented line will be characterized by its first and second intersection points x, y with M . Let $n(x)$ and $n(y)$ be the unit normal vectors to M at x and y .

Lemma 2.5.1. *One has: $\omega = d\lambda$ where*

$$\lambda = \frac{n(y)dx}{n(y) \cdot (y - x)} + \frac{n(x)dy}{n(x) \cdot (y - x)}.$$

Proof. A Finsler metric is characterized by its Lagrangian L , a positive function on the tangent bundle, homogeneous of degree one in the velocity vectors (see [Ar]). Introduce the following notation: q, u and p will denote points, tangent vectors and covectors in \mathbf{R}^n . The symplectic form in question is obtained from the symplectic form in $T^*\mathbf{R}^n$ by symplectic reduction. Using the Legendre transform $p = L_u$, one writes the Liouville form pdq as $L_u dq$. Restricting to the space of oriented lines, one obtains the desired 1-form λ .

Let q be a point on the line xy and u a tangent vector at q along xy . Let s, t be the reals such that $q = (1 - t)x + ty$, $u = s(y - x)$. Then x, y, t, s are coordinates in $T\mathbf{R}^n$.

First we compute the Lagrangian in these coordinates. Identify the segment (xy) with $(0, 1)$ by an affine transformation. Given a point $t \in (0, 1)$, consider the Hilbert geodesic $x(\tau)$ through t . Then $x(0) = t$, and the Hilbert distance $d(t, x(\tau)) = \tau$, that is,

$$\ln\left(\frac{(1-t)x(\tau)}{t(1-x(\tau))}\right) = \tau.$$

Solving for x , differentiating and evaluating at $\tau = 0$ one finds: $x'(0) = t(1-t)$. It follows that

$$|u|_{Hilbert} = \frac{|u|_{Euclidean}}{t(1-t)}.$$

Returning to the segment (xy) , one obtains the formula: $L(x, y, t, s) = s/t(1-t)$.

Next we compute L_u . One has:

$$dq = (1-t)dx + tdy + (y-x)dt, \quad du = (y-x)ds + s(dy-dx).$$

Write the differential dL in two ways:

$$L_u du + L_q dq = L_u(y-x)ds + L_u s(dy-dx) + (1-t)L_q dx + tL_q dy + L_q(y-x)dt.$$

One obtains the equations:

$$(y-x)L_u = \frac{1}{t(1-t)}, \quad (y-x)L_q = -\frac{s(1-2t)}{t^2(1-t)^2}, \quad (sL_u + tL_q)dy = 0, \quad (sL_u - (1-t)L_q)dx = 0.$$

The latter two 1-forms vanish only if the vectors in front of dy and dx are proportional to $n(y)$ and $n(x)$, respectively:

$$sL_u + tL_q = an(y), \quad sL_u - (1-t)L_q = bn(x).$$

Therefore

$$sL_u = a(1-t)n(y) + btn(x), \quad L_q = an(y) - bn(x).$$

Using the first two equations, one finds:

$$a = \frac{s}{(1-t)^2 n(y) \cdot (y-x)}, \quad b = \frac{s}{t^2 n(x) \cdot (y-x)}.$$

Thus

$$L_u = \frac{n(y)}{(1-t)n(y) \cdot (y-x)} + \frac{n(x)}{tn(x) \cdot (y-x)}.$$

Since $n(x)dx = 0$ and $n(y)dy = 0$, it follows that

$$L_u dq = \frac{n(y)dx}{n(y) \cdot (y-x)} + \frac{n(x)dy}{n(x) \cdot (y-x)} + d \ln\left(\frac{t}{1-t}\right),$$

and one may omit the last exact term to obtain the 1-form λ on the space of rays intersecting M .

Finally, consider the manifold C that consists of the rays through a fixed point inside M . Then C is obtained from a fiber of the cotangent bundle by symplectic reduction. The Liouville form vanishes on the fibers of $T^*\mathbf{R}^n$, therefore the form λ on C is cohomologous to zero.

We apply Lemma 2.5.1 to the case when M is the unit sphere. The Hilbert metric is the hyperbolic metric in the Beltrami-Klein model of \mathbf{H}^n . As it was explained in the Introduction, a transverse line field ξ along the sphere determines a submanifold L^{n-1} of the space of rays in \mathbf{H}^n .

Theorem 2.5.2. *The transverse line field ξ is exact if and only if L is an exact Lagrangian submanifold of the space of rays in \mathbf{H}^n .*

Proof. One has: $n(x) = x$, $n(y) = y$. Therefore $\lambda = (ydx - xdy)/(1-xy)$. Since $\lambda + d \ln(1-xy) = -2(xdy)/(1-xy)$ the form $\lambda|_L$ is exact if and only if so is $(xdy)/(1-xy)|_L$.

Let $v(y)$, $y \in M$ be a vector field generating a transverse line field along M , normalized by $yv(y) = 1$. Then the other intersection point of the line $\xi(y)$ with M is

$$x = y - 2 \frac{v(y)}{v(y) \cdot v(y)}.$$

It follows that $1 - xy = 2/v^2$. One has on L :

$$-\frac{xdy}{1-xy} = \frac{v^2}{2} \left(-ydy + \frac{2vdy}{v^2} \right) = v(y)dy,$$

the second equality due to the fact that $ydy = 0$. Thus L is exact Lagrangian if and only if the field ξ is exact.

We finish the section with a problem: describe the closed convex hypersurfaces M such that a transverse line field along M is exact if and only if the corresponding submanifold in the space of rays intersecting M is exact Lagrangian with respect to the form λ from Lemma 2.5.1. We conjecture that only ellipsoids have this property.

3. Projective billiards

3.1. Start with an equation describing the projective billiard reflection in a closed convex hypersurface $M \subset \mathbf{R}^n$ equipped with a transverse line field ξ . Let v be a vector field along ξ such that $vn = 1$, and let the ray xy reflect to the ray yz ; here $x, y, z \in M$.

Lemma 3.1.1. *One has:*

$$\frac{y-x}{n(y) \cdot (y-x)} + \frac{y-z}{n(y) \cdot (y-z)} = 2v(y).$$

Proof. The vector

$$\frac{y-x}{n(y) \cdot (y-x)} - \frac{y-z}{n(y) \cdot (y-z)}$$

is normal to $n(y)$, therefore this vector is tangent to M at y . Thus this vector spans the intersection line of the plane generated by $y-x$ and $z-y$ with the hyperplane $T_y M$. The fourth vector

$$\frac{y-x}{n(y) \cdot (y-x)} + \frac{y-z}{n(y) \cdot (y-z)}$$

constitutes a harmonic quadruple with the above three. Therefore

$$\frac{y-x}{n(y) \cdot (y-x)} + \frac{y-z}{n(y) \cdot (y-z)} = tv(y).$$

Take scalar product with $n(y)$ to conclude that $t = 2$.

This lemma implies the next result.

Theorem 3.1.2. *Let M be a closed quadratically convex hypersurface equipped with an exact transverse line field. For every two points A, B inside M there exist at least two rays through A that, after the projective billiard reflection in M , pass through B .*

Proof. Let $vdn = df$. Consider the function

$$h(y) = \ln(n(y) \cdot (y-A)) + \ln(n(y) \cdot (y-B)) - 2f(y).$$

This function on M has at least two critical points. One has:

$$dh = \frac{(y-A)dn + ndy}{n(y-A)} + \frac{(y-B)dn + ndy}{n(y-B)} - 2vdn = \left(\frac{y-A}{n(y-A)} + \frac{y-B}{n(y-B)} - 2v \right) dn.$$

The vector in front of dn in the last formula is orthogonal to n , and M is quadratically convex. Therefore if $dh(y) = 0$ then

$$\frac{y-A}{n(y) \cdot (y-A)} + \frac{y-B}{n(y) \cdot (y-B)} = 2v(y).$$

According to Lemma 3.1.1, this implies that the ray Ay reflects to the ray yB .

We conjecture that the quadratic convexity condition may be relaxed, and the result will hold for the hypersurfaces, star-shaped with respect to the points A and B .

3.2. It was mentioned in the Introduction that for every transverse line field the projective billiard transformation in an ellipse preserves the symplectic form on the space of rays, associated with the hyperbolic metric. In this section we prove that this property characterizes the ellipses.

Theorem 3.2.1. *Let M be a strictly convex closed plane curve equipped with a transverse line field. If the projective billiard transformation preserves the symplectic structure on the oriented lines, associated with the Hilbert metric inside M , then M is an ellipse.*

Proof. Give M an affine parameterization: this means that $[M'(t), M''(t)] = 1$ for every t where $[\cdot, \cdot]$ is the cross-product of vectors in the plane. A ray is characterized by its first and second intersection points $M(t_1)$ and $M(t_2)$ with the curve. We use (t_1, t_2) as coordinates in the space of rays.

First, we rewrite the symplectic form ω from Lemma 2.5.1 in these coordinates. If $x = M(t_1)$, $y = M(t_2)$ then $dx = M'(t_1)dt_1$, $dy = M'(t_2)dt_2$. Since

$$\frac{n(y) \cdot M'(t_1)}{n(y) \cdot (y - x)} = \frac{[M'(t_1), M'(t_2)]}{[M(t_2) - M(t_1), M'(t_2)]}, \quad \frac{n(x) \cdot M'(t_2)}{n(x) \cdot (y - x)} = \frac{[M'(t_2), M'(t_1)]}{[M(t_2) - M(t_1), M'(t_1)]},$$

the 1-form λ from Lemma 2.5.1 is as follows:

$$\lambda = \frac{[M'(t_1), M'(t_2)]}{[M(t_2) - M(t_1), M'(t_2)]} dt_1 + \frac{[M'(t_2), M'(t_1)]}{[M(t_2) - M(t_1), M'(t_1)]} dt_2.$$

Differentiating and using the identities:

$$[a, b][c, d] = [a, c][b, d] + [a, d][c, b] \quad \text{for all } a, b, c, d \quad (1)$$

and $[M'(t), M''(t)] = 1$, one concludes that

$$\omega = \left(\frac{[M(t_2) - M(t_1), M'(t_2)]}{[M(t_2) - M(t_1), M'(t_1)]^2} - \frac{[M(t_2) - M(t_1), M'(t_1)]}{[M(t_2) - M(t_1), M'(t_2)]^2} \right) dt_1 \wedge dt_2. \quad (2)$$

Let T be the projective billiard transformation, and let $T(t_1, t_2) = (t_2, t_3)$. Cross-multiply the equality of Lemma 3.1.1 by $v(y)$ where $y = M(t_2)$ and, as before, replace the dot product with $n(y)$ by the cross-product with $M'(t_2)$. One obtains the equality:

$$\frac{[M(t_2) - M(t_1), v(y)]}{[M(t_2) - M(t_1), M'(t_2)]} + \frac{[M(t_3) - M(t_2), v(y)]}{[M(t_3) - M(t_2), M'(t_2)]} = 0. \quad (3)$$

Take the exterior derivative, wedge multiply by dt_2 , simplify using (1), and cancel the common factor $[v(y), M'(t_2)]$ to arrive at the equality:

$$-\frac{[M(t_2) - M(t_1), M'(t_1)]}{[M(t_2) - M(t_1), M'(t_2)]^2} dt_1 \wedge dt_2 = \frac{[M(t_3) - M(t_2), M'(t_3)]}{[M(t_3) - M(t_2), M'(t_2)]^2} dt_2 \wedge dt_3. \quad (4)$$

If $T^*\omega = \omega$ then, according to (2), one has:

$$\left(\frac{[M(t_2) - M(t_1), M'(t_2)]}{[M(t_2) - M(t_1), M'(t_1)]^2} - \frac{[M(t_2) - M(t_1), M'(t_1)]}{[M(t_2) - M(t_1), M'(t_2)]^2} \right) dt_1 \wedge dt_2 =$$

$$= \left(\frac{[M(t_3) - M(t_2), M'(t_3)]}{[M(t_3) - M(t_2), M'(t_2)]^2} - \frac{[M(t_3) - M(t_2), M'(t_2)]}{[M(t_3) - M(t_2), M'(t_3)]^2} \right) dt_2 \wedge dt_3.$$

According to (4), the second term at the left hand side equals the first one at the right.

Thus

$$\frac{[M(t_2) - M(t_1), M'(t_2)]}{[M(t_2) - M(t_1), M'(t_1)]^2} dt_1 \wedge dt_2 = - \frac{[M(t_3) - M(t_2), M'(t_2)]}{[M(t_3) - M(t_2), M'(t_3)]^2} dt_2 \wedge dt_3. \quad (5)$$

Divide (4) by (5) and take the cubic root to arrive at the equality

$$\frac{[M(t_2) - M(t_1), M'(t_1)]}{[M(t_2) - M(t_1), M'(t_2)]} = \frac{[M(t_3) - M(t_2), M'(t_3)]}{[M(t_3) - M(t_2), M'(t_2)]}. \quad (6)$$

Change the notation: let $t_2 = t$, $t_3 = t + \epsilon$, $t_1 = t - \delta$. Denote the right hand side of (6) by $f(t, \epsilon)$; then (6) reads:

$$f(t - \delta, \delta) = \frac{1}{f(t, \epsilon)}. \quad (7)$$

Differentiating the equality $[M'(t), M''(t)] = 1$, one has: $[M'(t), M'''(t)] = 0$. Thus $M'''(t) = -k(t)M'(t)$ where the function $k(t)$ is called the affine curvature. Further differentiating, one finds:

$$[M', M^{IV}] = -k, \quad [M'', M'''] = k, \quad [M', M^V] = -2k', \quad [M'', M^{IV}] = k'. \quad (8)$$

We will show that $k' = 0$ identically. This will imply the result since the only curves with constant affine curvature are the conics.

Using the Taylor expansion up to the fifth derivatives and taking (8) into account, one finds:

$$f(t, \epsilon) = 1 + \frac{\epsilon^3 k'(t)}{60} + O(\epsilon^4), \quad (9)$$

and hence

$$f(t - \delta, \delta) = 1 + \frac{\delta^3 k'(t - \delta)}{60} + O(\delta^4). \quad (10)$$

Clearly $\delta = O(\epsilon)$. Consider the Taylor expansion of (3) up to ϵ^2 :

$$\frac{[\delta M' - \frac{\delta^2}{2} M'', v]}{[\delta M' - \frac{\delta^2}{2} M'', M']} + \frac{[\epsilon M' + \frac{\epsilon^2}{2} M'', v]}{[\epsilon M' + \frac{\epsilon^2}{2} M'', M']} = 0.$$

It follows that $\delta = \epsilon + O(\epsilon^2)$. Then (10) and (9) imply that

$$f(t - \delta, \delta) = 1 + \frac{\epsilon^3 k'(t)}{60} + O(\epsilon^4) = f(t, \epsilon),$$

and one concludes from (7) that $k'(t) = 0$ for all t .

3.3. Let M be a circle in the plane and ξ a transverse line field along M . The phase space of the projective billiard map T is a cylinder C , and T preserves the area form $\omega = d\lambda$ from Theorem 2.5.2.

Lemma 3.3.1. *The flux of T equals zero if and only if ξ is exact.*

Proof. Let $L \subset C$ be the curve that consists of the lines $\xi(x)$, $x \in M$, oriented outward, and let τ be the involution of C that changes the orientation of a line to the opposite. Then $T(L) = \tau(L)$. The flux of T is the signed area between the curves L and $T(L)$, that is,

$$\int_L \lambda - \int_{\tau(L)} \lambda.$$

Clearly, $\tau^*(\lambda) = -\lambda$. Hence the flux of T equals $2 \int_L \lambda$. According to Theorem 2.5.2 this integral vanishes if and only if ξ is exact.

An invariant circle is a simple closed noncontractible T -invariant curve $\Gamma \subset C$. The map T is a twist map of the cylinder, the leaves of the vertical foliation consisting of the outward oriented lines through a fixed point on M . According to Birkhoff's theorem, an invariant circle intersects each vertical leaf once (see, e.g., [H-K]). Γ is a one parameter family of lines, and their envelope $\gamma \subset \mathbf{R}^2$ is called a projective billiard caustic. The caustic may have cusp singularities but from every point $x \in M$ there are exactly two tangents to γ . A ray, tangent to γ , remains tangent to it after the projective billiard reflection.

Corollary 3.3.2. *If a projective billiard map inside a circle has an invariant circle then the corresponding transverse line field is exact.*

In other words, let γ be a closed curve inside the circle M such that from every point $x \in M$ there are exactly two tangents to γ . Let $\xi(x)$ be the line that constitutes a harmonic quadruple with these two tangents and the tangent line to M at x . Then the transverse line field ξ is exact.

This property is specific to circles: we will construct a pair (M, γ) such that the corresponding transverse field ξ will fail to be exact.

Example. The caustic γ will be a triangle $A_1A_2A_3$ (if one wishes one may smoothly round the corners). Let M be a closed strictly convex curve enclosing the triangle. Denote by P_i and Q_i , $i = 1, 2, 3$ the intersections of the rays A_iA_{i+1} and $A_{i+1}A_i$ with M (the indices are considered modulo 3). For example, a ray through the point A_3 reflects in the arc Q_1P_3 to a ray through A_2 . This determines the transverse line field ξ along Q_1P_3 , and a similar consideration applies to the other five arcs of M .

According to Theorem 1.2.4, the field ξ is exact if and only if it is generated by the acceleration vectors $M''(t)$ for some parameterization $M(t)$. Consider again the arc Q_1P_3 . The desired parameterization is determined by the equation (3) from the proof of Theorem 3.2.1. Namely, setting $v = M''(t)$, $M(t_1) = A_2$, $M(t_3) = A_3$ and $M(t_2) = M(t)$, this equation can be rewritten as

$$([M(t) - A_2, M'(t)][M(t) - A_3, M'(t)])' = 0.$$

Thus $[M - A_2, M'][M - A_3, M']$ is constant. Similar equations determine the parameterizations of the other five arcs. One can make a consistent choice of the constants involved if and only if

$$\prod_{i=1}^3 \frac{[P_i - A_i, M'|_{P_i}][Q_i - A_i, M'|_{Q_i}]}{[P_i - A_{i+1}, M'|_{P_i}][Q_i - A_{i+1}, M'|_{Q_i}]} = 1.$$

This condition is equivalent to

$$\frac{|A_1P_1||A_2P_2||A_3P_3||A_1Q_1||A_2Q_2||A_3Q_3|}{|A_2P_1||A_3P_2||A_1P_3||A_2Q_1||A_3Q_2||A_1Q_3|} = 1.$$

Clearly this equality does not hold for a generic curve M (but it holds for a circle or an ellipse as follows from elementary geometry). It is interesting to remark that a similar construction will not work for a "two-gon": it is shown in [T 2] that for every two points A, B inside M there is an exact transverse line field along M such that every ray through A projectively reflects in M to a ray through B .

We finish this section with a description of a dynamical system on the one-sheeted hyperboloid, equivalent to the projective billiard in a circle.

Consider 3-space with the Lorentz quadratic form $Q = x^2 + y^2 - z^2$; denote by H_+ the upper sheet of the hyperboloid $Q = -1$, by H_0 the cone $Q = 0$ and by H_- the one-sheeted hyperboloid $Q = 1$. The Lorentz metric, restricted to H_+ , is a metric of constant negative curvature. The projection from the origin to the plane $z = 1$ takes H_0 to the unit circle and identifies H_+ with the Beltrami-Klein model of the hyperbolic plane. In particular, the straight lines on H_+ are its intersections with the planes through the origin.

The Q -orthogonal complement to such a plane is a line, intersecting H_- . Taking orientation into account, to every oriented line in H_+ there corresponds a point of H_- , and the orientation reversing involution on the lines corresponds to the central symmetry of H_- . This duality with respect to the quadratic form Q identifies H_- with the space of rays in the hyperbolic plane, that is, the phase space of the projective billiard in a circle.

The hyperboloid H_- carries two families of straight lines called the rulings; denote these families by U and V . Given a transverse line field ξ along the circle, consider it as

a family of outward oriented lines; this determines a noncontractible simple closed curve $\gamma \subset H_-$. For example, if ξ consists of the lines through a fixed point then γ is a plane section of the hyperboloid. A field ξ is exact if and only if the signed area between γ and the equator of H_- equals zero.

Let T be the projective billiard transformation considered as a map of H_- .

Theorem 3.3.3. *The curve γ intersects each ruling from one of the families, say, U , at a unique point. Given a point $x \in S$, let $u \in U$ be the ruling through x . Then $T(x)$ is obtained from x by the composition of the two reflections: first, in the intersection point of u with γ , and then in the origin.*

Proof. Let u be a ruling of H_- . Consider the plane π through u and the origin. Since H_0 and H_- are disjoint, the intersection of π with H_0 is a line l , parallel to u (and not a pair of lines). It follows that π is tangent to C . If l is a line on the cone H_0 then its Q -orthogonal complement is a plane tangent to H_0 along l , that is, coincides with π . The line l projects to a point y of the unit circle. It follows that the duality takes a line through y to a point on u .

Taking orientations into account, we obtain the following result: the one parameter families of incoming and outgoing rays through a point y on the unit circle correspond on H_- to two parallel rulings $u \in U$ and $v \in V$. Then $\xi(y)$, oriented outward, corresponds to the unique intersection point of γ with u . The tangent line to the unit circle at y corresponds to the point of u at infinity.

Consider the projective billiard reflection at y . The incoming and outgoing lines, both oriented outward, correspond to the points $x, x_1 \in u$. These points, along with the point $\gamma \cap u$ and the point of u at infinity, constitute a harmonic quadruple. Thus x and x_1 are symmetric with respect to the point $\gamma \cap u$. Reversing the orientation of the outgoing line amounts to reflecting x_1 in the origin, and the result follows.

Remark. The canonical area form on the space of rays in the hyperbolic plane is the standard area form on H_- induced by the Euclidean metric. Equivalently, this area equals the volume of the layer between H_- and an infinitely close homothetic surface. The axial projection of H_- to the cylinder $x^2 + y^2 = 1$ is area preserving (a similar fact for the sphere is due to Archimedes).

3.4. In this section we will derive the mirror equation for the projective billiard in the unit circle. A similar approach works for an arbitrary billiard curve but the resulting formula is rather ugly.

Let $M(t) = (\cos t, \sin t)$ be the unit circle, and let $\phi(t)$ be the angle made by the line of the transverse line field at $M(t)$ with the tangent vector $M'(t)$. Let A, B be two

points inside M such that the infinitesimal beam of rays from A reflects in $M(t_0)$ to the infinitesimal beam of rays through B . Let $\alpha(t)$ and $\beta(t)$ be the angles made by the rays $M(t)A$ and $M(t)B$ with $M'(t)$, and let $|M(t)A| = a(t)$, $|M(t)B| = b(t)$. Omit the argument for the values of these functions at t_0 .

Theorem 3.4.1. *One has:*

$$\frac{2\phi'}{\sin^2 \phi} + \frac{1}{\sin^2 \alpha} + \frac{1}{\sin^2 \beta} = \frac{1}{a \sin \alpha} + \frac{1}{b \sin \beta}.$$

Proof. It is easy to see that if the ray $AM(t)$ reflects to the ray $M(t)B$ then $2 \cot \phi(t) = \cot \alpha(t) + \cot \beta(t)$. Differentiating and evaluating at t_0 one obtains the equation:

$$\frac{2\phi'}{\sin^2 \phi} = \frac{\alpha'}{\sin^2 \alpha} + \frac{\beta'}{\sin^2 \beta}.$$

The result will follow once we show that

$$\alpha' = \frac{\sin \alpha}{a} - 1, \quad \beta' = \frac{\sin \beta}{b} - 1.$$

Indeed, $A - M(t) = a(-\sin(t + \alpha), \cos(t + \alpha))$. Differentiating, cross-multiplying by the vector $(-\sin(t + \alpha), \cos(t + \alpha))$ and evaluating at t_0 yields: $\sin \alpha = a(1 + \alpha')$. A similar computation for β finishes the proof.

We conclude the paper with a conjecture on the integrability of projective billiards. The projective billiard inside a plane curve M will be called integrable near the boundary if there is a neighbourhood of M foliated by the caustics. For example, this holds for the usual billiard inside an ellipse, the caustics being the confocal ellipses. Birkhoff's conjecture states that if a plane billiard is integrable near the boundary then the billiard curve is an ellipse; as far as I know, this conjecture is still open. A related theorem by Bialy states that if the whole phase space of a plane billiard is foliated by invariant circles then the billiard curve is a circle – see [Bi].

Recall that a pencil of conics consists of the conics, passing through fixed four points (in what follows these points will be complex).

Conjecture 3.4.2. *a). If the projective billiard inside a closed convex smooth plane curve M is integrable near the boundary then M and the caustics are the ellipses belonging to the family, projectively dual to a pencil of conics.*

b). If the whole phase space of a plane projective billiard is foliated by invariant circles then M is an ellipse and the transverse field consists of the lines through a fixed point inside M .

As a justification, we prove the converse statements. The second one is obvious: modulo a projective transformation, M is a circle, and the point is its center. Then the concentric circles are the caustics.

To prove the statement converse to a), consider the curve M^* , projectively dual to M . A point $x \in M$ becomes a tangent line l to M^* , and the incoming and outgoing rays at x , tangent to a caustic C , become the intersection points of l with the dual curve C^* . The curves C^* constitute a pencil of conics, and one wants to show that the (local) involution of l that interchanges its intersection points with each of these conics is a projective transformation. According to a Desargues theorem this holds for every line intersecting a nondegenerate pencil of conics (see [Be]), and we are done.

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