two matrix we deduce

$$\boxed{B_{\mathbb{R}^n \oplus \mathbb{R}, (p, \lambda)}(\hat{L}) = B_{M,p}(f|_M) \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},}$$

that is, the bordered Hessian is congruent to the constrained Hessian summed with a saddle point pair. This is the main result in **[1]**. Applying it to the above example thus requires the diagonalisation of a four by four matrix which, although not difficult to do, is rather tiresome. It is worth emphasising again at this point the importance of the distinction between Hessian matrices and bilinear forms. It is the condition $\nabla\hat{L}(p, \lambda) = 0$ which allows us to use bilinear forms to bridge the gap between the above rather special coordinates and the coordinates we are given in the example. Otherwise it would not be clear that our four by four matrix, diagonalised or not, had anything to do with the local behaviour of $f|_M$.

5. CONCLUSION. We have presented two methods for discerning the nature of a constrained critical point. We can construct a basis for the tangent space of the submanifold and diagonalise the $(n - 1) \times (n - 1)$ restricted Hessian matrix. Alternatively, we can diagonalise the $(n + 1) \times (n + 1)$ bordered Hessian matrix. Which of these two methods is to be preferred is perhaps a matter of taste, although in the above example the former turned out to be the easier.

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A Cone Eversion

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In the punctured plane $\mathbf{R}^2 - (0,0)$ two functions are given: $f_0(x, y) = \sqrt{x^2 + y^2}$ and $f_1(x, y) = -\sqrt{x^2 + y^2}$. Their gradients are the constant radial vector fields (Fig. 1). Certainly, these fields are homotopic as nondegenerate vector fields (that is, they can be included into a continuous one-parameter family of vector fields without zeroes in the punctured plane): just rotate each vector through 180°. Can one perform such a homotopy in the class of nondegenerate *gradient* vector fields? In other words, can one include the functions $f_0(x, y)$ and $f_1(x, y)$ into a

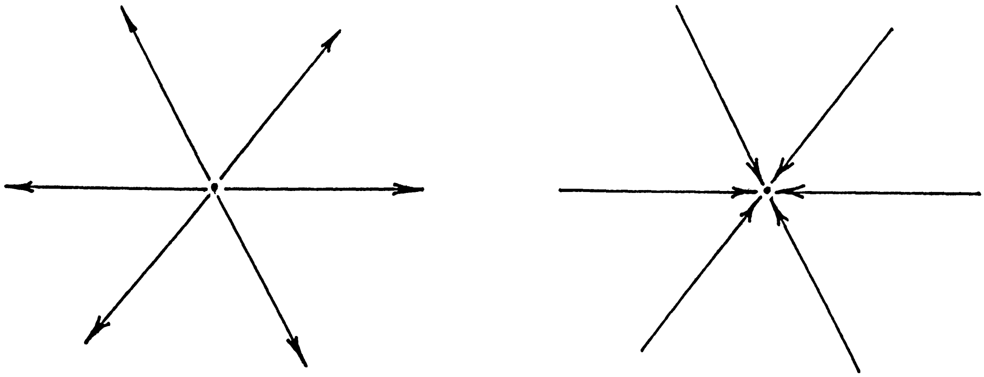

Figure 1.

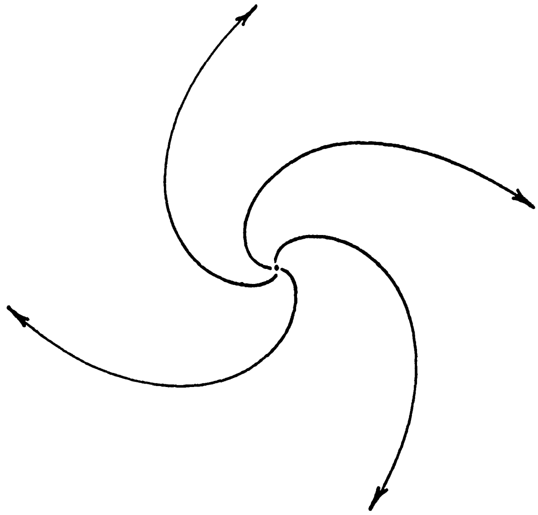

Figure 2.

one-parameter family of smooth functions $f_t(x, y)$ without critical points in the punctured plane, continuously depending upon the parameter t? The reader is encouraged to make his/her own attempt (warning: rotation of the vectors in the same sense does not work—the field in Fig. 2 is not a gradient!)

Formulate the problem more geometrically. Fix an open annulus in the horizontal x,y-plane, say, the annulus $1 < \sqrt{x^2 + y^2} < 3$. It is diffeomorphic to the punctured plane, so it is enough to solve the problem in the annulus (indeed, if one has a family of functions f_t without critical points in the annulus, composing it with the diffeomorphism yields a desired family of functions in the punctured plane). Consider the function $f_t(x,y)$ as the height function of a surface S_t in space, whose projection onto the horizontal plane is the fixed annulus. The surfaces S_0 and S_1 are cones (Fig. 3); the latter is the "lamp" and the former—the "lump" (in analogy with the "cap" $\cap$ and the "cup" $\cup$). What one wants to achieve is a deformation of the "lump" S_0 to the "lamp" S_1, so that each intermediate surface S_t does not have a horizontal tangent plane at any point.

Figure 3.

Here is an example of the deformation in question. The surface S_t is given by the equation

$$z = g_t(\alpha) + 0.25(r - 2)h_t(\alpha)$$

in the cylindrical coordinates (α, r, z); here $0 \le \alpha \le 2\pi$, $1 < r < 3$ and the "time" parameter t varies from 0 to 4. The functions g and h are:

$$g_t = t \sin \alpha, \qquad h_t = (1 - t) + t(0.5 + \cos \alpha); \qquad t \in [0,1];$$

$$g_t = (2 - t)\sin \alpha + (t - 1)\sin 2\alpha, \qquad h_t = \cos \alpha + 0.5(2 - t); \qquad t \in [1,2];$$

$$g_t = -(t - 2)\sin \alpha + (3 - t)\sin 2\alpha, \qquad h_t = \cos \alpha - 0.5(t - 2); \qquad t \in [2,3];$$

$$g_t = -(4 - t)\sin \alpha, \qquad h_t = -(t - 3) + (4 - t)(\cos \alpha - 0.5); \qquad t \in [3,4].$$

The reader may (but probably will not) verify that, for all values of t, the function $z_t(\alpha, r)$ does not have critical points.

One could stop here; but I believe that I owe the reader some explanations. First, the existence of the homotopy in question is a very particular consequence of the Gromov-Phillips theorem and the Gromov h-principle theory (see **[G]** and **[H]**), The proofs in this theory are by no means constructive, so explicit constructions are of interest. A famous example is turning a sphere inside out—another consequence of the Gromov theory (more precisely, of its predecessor, the Hirsch-Smale theorem); see, e.g., **[Fr]** or the movie under preparation at the Minnesota Geometry Center. The problem we are concerned with here was mentioned in **[F]** (and was given to me by my advisor D. Fuchs some 17 years ago; I believe the present construction is similar to a somewhat obscure one I produced then).

Secondly, I should like to explain how the above formulas came up. Since the original and the terminal functions are linear in r, it is natural to look for the function z_t in the form:

$$z_t(\alpha, r) = g_t(\alpha) + \epsilon(r - 2)h_t(\alpha),$$

where g and h are periodic functions and ϵ is a small parameter to be chosen. The original "lump" surface corresponds to $g_0(\alpha) = 0$ and $h_0(\alpha) = const > 0$; the terminal "lamp"—to $g_4(\alpha) = 0$ and $h_4(\alpha) = const < 0$. It might be instructive to think about the surface S_t as a sort of closed rope ladder in space, whose axis is the curve

$$z = g_t(\alpha), \qquad 0 \le \alpha \le 2\pi, \qquad r = 2,$$

and whose rungs are the radial segments

$$z = g_t(\alpha) + \epsilon(r - 2)h_t(\alpha), \qquad \alpha = const, \qquad 1 < r < 3$$

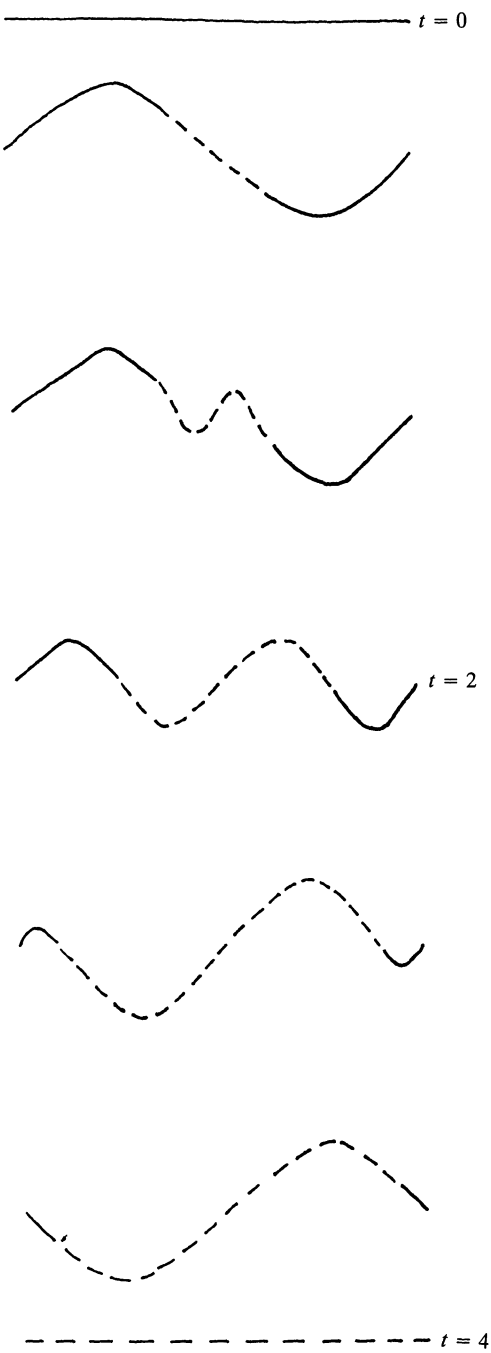

Figure 4.

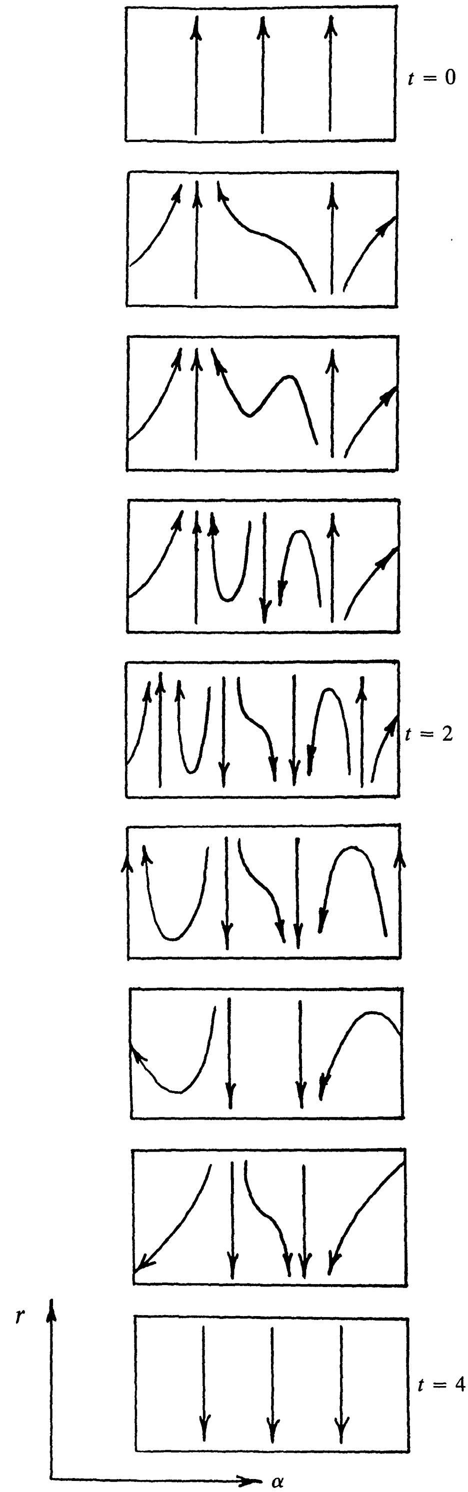

Figure 5.

with the slope of $\epsilon h_t(\alpha)$. So, at the beginning, the axis is a horizontal circle and the slopes of all the rungs are positive. At the end, the axis is again a horizontal circle, but the slopes of the rungs are all negative.

What one wants to avoid in the deformation are any points where the axis and the rungs are simultaneously horizontal. Thus the functions

$$\frac{d g_t(\alpha)}{d\alpha} + \epsilon(r-2)\frac{d h_t(\alpha)}{d\alpha} \quad \text{and} \quad h_t(\alpha)$$

should not have common zeros. If, for some t, the zeros of

$$\frac{d g_t(\alpha)}{d\alpha} \quad \text{and} \quad h_t(\alpha)$$

are disjoint, then so are the zeroes of

$$\frac{d g_t(\alpha)}{d\alpha} + \epsilon(r-2)\frac{d h_t(\alpha)}{d\alpha} \quad \text{and} \quad h_t(\alpha)$$

for a sufficiently small ϵ (use continuity and compactness of the circle). By compactness of the t-interval this ϵ can be chosen uniformly for all $t \in [0,4]$.

The strategy is clear now. First, change the shape of the axis of the rope ladder (i.e., the graph of $g(\alpha)$) into a non-horizontal curve, after which one can safely change the slope of the rungs (the sign of $h(\alpha)$) from positive to negative on its non-horizontal segments.

The graphs of $g_t(\alpha)$ are sketched in Fig. 4. The graphs are drawn in solid or broken lines; the former means that $h_t(\alpha)$ is positive, and the latter—that it is negative at the corresponding points α. The half-way picture ($t = 2$) is symmetric with respect to the time eversion: $t \to 2 - t$; from that point on one just repeats the process backwards (should one call this half-way surface S_2 the "limp"?). The reader is encouraged to use his/her favorite software to visualize the "limp" S_2. Fig. 5 shows the corresponding homotopy of the gradient vector fields, thus answering the original question.

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Answer to Picture Puzzle
(p. 22)

Cleve Moler, the principal creator of MATLAB.