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# ON POLYGONAL DUAL BILLIARD IN THE HYPERBOLIC PLANE*

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We study the polygonal dual billiard map in the hyperbolic plane. We show that for a class of convex polygons called *large* all orbits of the dual billiard map escape to infinity. We also analyse the dynamics of the dual billiard map when the dual billiard table is a regular polygon with all right angles.

## 1. Introduction and formulation of results

Given a compact convex plane domain $P$, one defines the dual billiard transformation $F$ of its exterior as follows. Let $x$ be a point outside of $P$. There are two support lines to $P$ through $x$; choose one of them, say, the right one from $x$'s view-point, and define $F(x)$ to be the reflection of $x$ in the support point. This definition applies if the support point is unique; otherwise $F(x)$ is not defined.

The dual billiard map is an outer counterpart of the usual billiard ball map, and it is also known as the "outer billiard". To the best of our knowledge, the dual billiard system was introduced by Bernhard Neumann [14] [1]; see also an earlier paper [4] in which the dynamical aspects are not discussed. J. Moser put forward the study of dual billiards in the framework of KAM theory in [12], and since then, the literature on dual billiards has continued to grow — see references at the end of this paper.

We are concerned with polygonal dual billiard tables $P$. The dual billiard map $F$ or its inverse $F^{-1}$ are not defined on the extensions of the sides of $P$. The singularity set $\Delta$ of the dual billiard map consists of the points $x$ such that $F^i(x)$ is not defined for some $i \in \mathbf{Z}$; in other words, $\Delta$ is the union of the images and preimages under $F$ of the lines that contain the sides of $P$. The set $\Delta$ is a countable union of segments and lines; it has zero measure. In the complement of $\Delta$, the map $F$ is a piecewise isometry.

We briefly survey what is known about polygonal dual billiards in the Euclidean plane. J. Moser asked in [13] whether the orbits of the polygonal dual billiard map can escape to infinity. A motivation comes from the case when the boundary of the dual billiard table is strictly convex and sufficiently smooth: then every orbit is confined to a compact domain bounded by a KAM invariant curve of the dual billiard map. Moser's question is still open for general polygons $P$. However there is a class of polygons, called *quasirational*, for which every orbit stays bounded.

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[1]We are grateful to K. Burns for this reference

Quasirational polygons are defined as follows. Let $l$ be a line through the origin in the plane. Consider the support lines $l_1$ and $l_2$ to $P$, parallel to $l$. Let $x_1$ and $x_2$ be the support points (uniquely defined unless $l$ is parallel to a side of $P$). Let $m(l)$ be the line $x_1x_2$ and $d(l) = |x_1x_2|$: see figure 1. Then $m(l)$ and $d(l)$ are piecewise constant functions of the line $l$. Assign the direction $m(l)$ to every point of the line $l$. This gives a homogeneous field of directions in the plane. One can show that the trajectories of this field are centrally symmetric convex $2k$-gons.

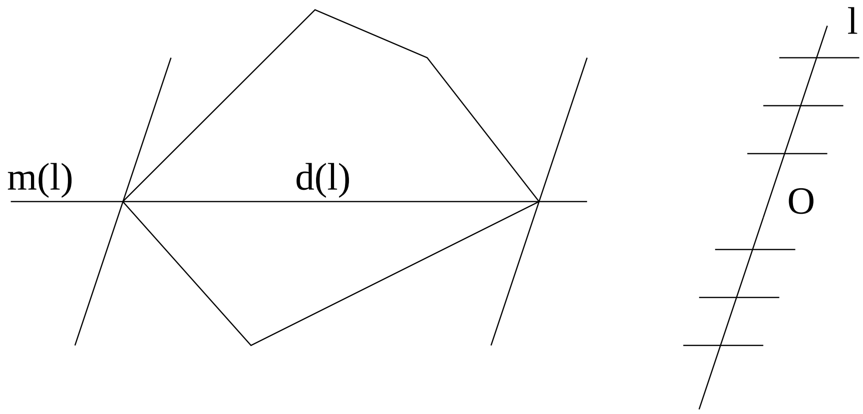


Fig. 1. Defining quasirational polygons

Let $Q$ be one of these polygons (uniquely defined, up to a dilation) and consider one of its sides; it has the direction $m(l)$ where $l$ is a line. Assign to this side the ratio of its length to $d(l)$. One obtains a collection of $k$ numbers $(t_1, \ldots, t_k)$, defined up to a common factor: $(t_1 : ... : t_k) \in \mathbf{RP}^{k-1}$. The polygon $P$ is called quasirational if $(t_1 : ... : t_k) \in \mathbf{QP}^{k-1}$, that is, if $Q$ can be rescaled so that $t_i \in \mathbf{Z}$ for all $i$.

The meaning of the described construction is as follows. Viewed from far away, the evolution of a point under the map $F^2$ appears as a continuous motion along a polygon, homothetic to $Q$. For example, if $P$ is a triangle then $Q$ is a centrally symmetric hexagon, and if $P$ is a square then $Q$ is a square, rotated through $\pi/4$ — see figure 2. Moreover, the speed of the limiting continuous motion along a side of $Q$ is $d(l)$ (again, up to a common factor). Therefore the numbers $t_i$ are the amounts of time it takes the moving point to traverse the sides of $Q$; see [18] for details.

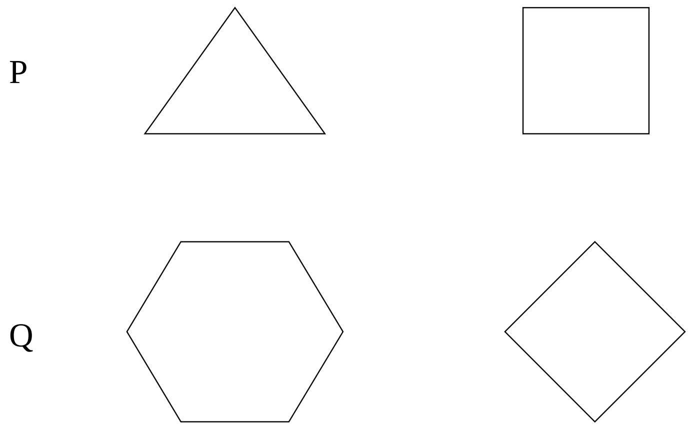


Fig. 2. Trajectories of the dual billiard map "at infinity"

**Theorem 1.** [8, 11, 15] *If $P$ is a quasirational polygon in the Euclidean plane then all orbits of the dual billiard map are bounded.*

Quasirational polygons include lattice polygons, that is, polygons whose vertices are lattice points. Therefore, for a lattice polygon, the dual billiard orbits are bounded, and being also discrete, each orbit is finite. We do not know a proof of this result bypassing Theorem 1. Another example of quasirational polygons are regular $n$-gons. Unless $n = 3, 4, 6$, a regular $n$-gon is not lattice, and the respective dual billiard map has infinite orbits — see figure 3 where the closure of an infinite orbit is shown for $n = 5$ (cf. [16, 17, 19]). Similar dynamics was recently described in [1, 5, 9] in the study of piecewise affine maps in dimension two.

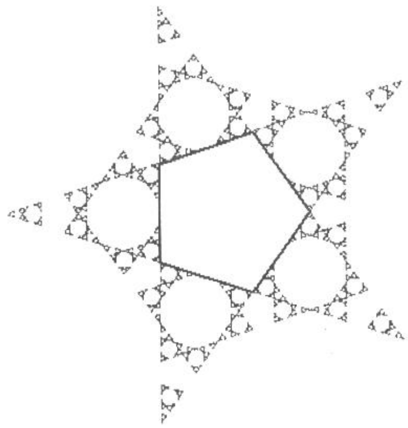

Fig. 3. Closure of an infinite dual billiard orbit for a regular pentagonal table

In this paper we make the first steps in the study of polygonal dual billiards in the hyperbolic plane. The map is defined in the same way as in the Euclidean plane but all the notions (straight line, distance, etc.) are understood in terms of hyperbolic geometry. In particular, the singularity set $\Delta$ is defined similarly to the Euclidean case.

We mostly use the Klein-Beltrami (or projective) model of hyperbolic geometry. The hyperbolic plane is represented by the open unit disc $D$ bounded by the unit circle $S^1$ ("circle at infinity"), straight lines — by chords of this circle, and the distance between points $x$ and $y$ is given by the formula

$$d(x, y) = \frac{1}{2} \ln[a, x, y, b]$$

where $a$ and $b$ are the intersection points of the line $xy$ with the circle and

$$[a, x, y, b] = \frac{(y - a)(b - x)}{(x - a)(b - y)}$$

is the cross-ratio. Isometries of the hyperbolic plane are represented by projective transformations of the plane, preserving $D$. Some pictures are drawn in the Poincaré disc model in which the hyperbolic plane is represented by $D$ but straight lines are arcs of circles, perpendicular to $S^1$. On one occasion we use the upper half-plane model too.

We address Moser's question: can orbits of the polygonal dual billiard map escape to infinity?

In the hyperbolic setup, the dual billiard map extends to the circle at infinity as a continuous circle map $f$. The map $f$ is a piecewise projective transformation of $S^1$ (identified, in the usual way, with the projective line); a point $x \in S^1$ is called a smooth point if it does not belong to the extension of a side of $P$. Let $\rho(P) \in (0, 1/2)$ be the rotation number of $f$. For an $n$-gon $P$, we show that $\rho(P) \geqslant 1/n$. A convex $n$-gon $P$ is called *large* if $\rho(P) = 1/n$ and the map $f$ has a hyperbolic $n$-periodic orbit (this orbit automatically consists of smooth points). The set of large polygons is open in the natural topology in the space of $n$-gons. An example of a large polygon is shown in figure 4.

Our main result is as follows.

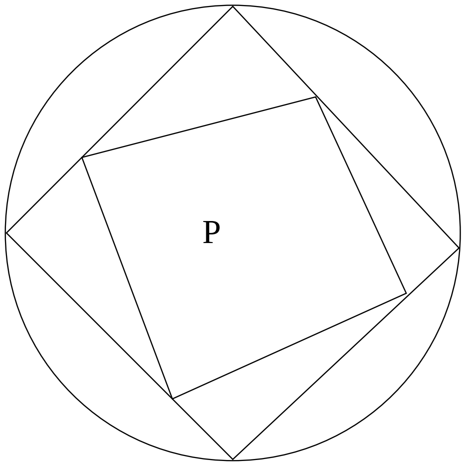


Fig. 4. Large quadrilateral

**Theorem 2.** *If $P$ is a large polygon then all orbits of the dual billiard map escape to infinity.*

Theorem 2 shows that the situation in the hyperbolic plane is very different from that in the Euclidean one. For example, many tiling polygons $P$ are large. For such polygons, the orbits of the dual billiard map are discrete and escape to infinity, in sharp contrast with the Euclidean case.

A related result is the next criterion for a triangle to be large. Let $a_i,\ i = 1, 2, 3$ be the sides of a triangle, $s = (a_1 + a_2 + a_3)/2$ its semiperimeter, $\alpha_i$ the angle opposite the $i$-th side, $h_i$ the altitude dropped on the $i$-th side, $K$ the area and $r$ is the radius of the incircle. We understand the index $i$ cyclically, so that $i + 3 = i$.

**Theorem 3.** *The triangle is large if and only if $H > 1$ where $H$ is given by any of the following equal expressions:*

$$H = \sinh h_i\ \sinh a_i = \sin \alpha_i\ \sinh a_{i+1}\ \sinh a_{i+2} = 2 \sinh s\ \tanh r =$$

$$2\sqrt{\sinh s\ \sinh(s - a_1)\ \sinh(s - a_2)\ \sinh(s - a_3)} =$$

$$4 \sin(K/2)\ \cosh(a/2)\ \cosh(b/2)\ \cosh(c/2)$$

*for all $i = 1, 2, 3$.*

An equivalent condition, in terms of the angles, reads:

$$\cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3 + 2 \cos \alpha_1 \cos \alpha_2 \cos \alpha_3 > 1 + \sin \alpha_1 \sin \alpha_2 \sin \alpha_3.$$

A triangle tiles the hyperbolic plane if its angles are $2\pi/p, 2\pi/q, 2\pi/r$ with $1/p + 1/q + 1/r < 1/2$. Up to isometries, all but 532 tiling triangles are large.

Theorems 2 and 3 are proved in Section 2, along with various properties of large polygons.

Another result of this paper is a detailed study of the dual billiard map in the case when $P$ is a tiling regular $n$-gon with right angles ($n \geqslant 5$). Our results are summarized in Theorem 4, Section 3. We prove that all orbits of the dual billiard map are finite, and the map decomposes into a countable number of transitive cyclic permutations of finite sets. We compute the respective rotation numbers and show that

$$\rho(P) = \frac{n - \sqrt{n(n-4)}}{2n}$$

(in a sense, this formula holds for $n = 4$ as well: a square tiles the Euclidean, not the hyperbolic, plane, and the dual billiard map "at infinity" is just a central symmetry with the rotation number 1/2).

We do not know whether there exist polygons in the hyperbolic plane for which all orbits of the dual billiard map are bounded but not all orbits are finite. Such polygons would be analogs of quasirational, but not lattice, polygons in the Euclidean setup.

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## 2. Large polygons

In this section we study properties of large polygons and prove Theorems 2 and 3. We use the notation from Introduction: $F$ or $F_P$ is the dual billiard map associated with a polygon $P$ and $f$ or $f_P$ is the extension of $F$ to the circle at infinity; $\rho(P)$ is the rotation number of $f$, the singularity set of $F$ is denoted by $\Delta$, etc. Start with a simple observation.

**Lemma 1.** *Let $P \subset Q$ be two convex polygons in the hyperbolic plane. Then $\rho(P) \geqslant \rho(Q)$.*

*Proof.* Let $\tilde{f}_P$ and $\tilde{f}_Q$ be liftings of the circle maps to maps of $\mathbf{R}$. Then $\tilde{f}_P(x) \geqslant \tilde{f}_Q(x)$ for all $x \in \mathbf{R}$, and the result follows. ■

**Remark 2.** It well may be that $\rho(P) = \rho(Q)$ for a proper inclusion $P \subset Q$. The rotation number $\rho(P)$ depends continuously on $P$, and for generic polygons, this function is a "devil's staircase": it assumes rational values on open sets in the space of polygons. This is a well known mode locking phenomenon for circle maps — see, e.g., [10].

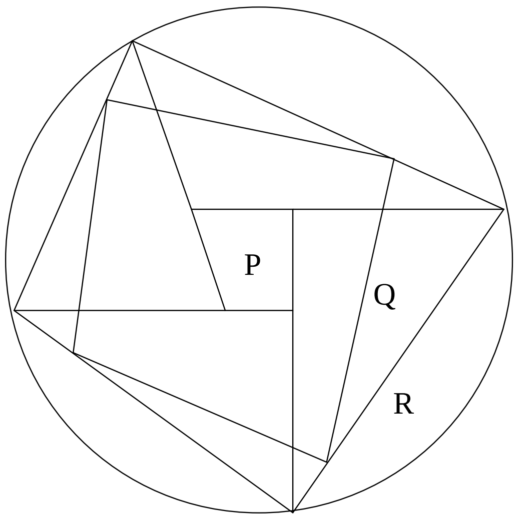


Fig. 5. Proving that $\rho(P) \geqslant 1/n$

**Proposition 3.** *Let $P$ be a convex $n$-gon in the hyperbolic plane. Then one has: $\rho(P) \geqslant 1/n$.*

*Proof.* Extend the sides of $P$ in the clockwise (negative) direction until they intersect $S^1$. The intersection points are the vertices of a convex inscribed $n$-gon $R$. Pick a point on each side of $R$, sufficiently close to the vertices, to obtain a convex $n$-gon $Q$ containing $P$ — see figure 5. The vertices of $R$ constitute an $n$-periodic orbit of $f_Q$ with the rotation number $1/n$. By Lemma 1, $\rho(P) \geqslant \rho(Q) = 1/n$. ■

Let $P$ be a convex $n$-gon with $\rho(P) = 1/n$. Then $f$ has an $n$-periodic orbit. Consider such an orbit and connect its consecutive points to obtain a convex polygon $R$. The polygon $P$ is inscribed into $R$.

**Lemma 4.** *Every $n$-periodic orbit of $f$ consists of smooth points, and there are at most two such orbits.*

*Proof.* Since both $P$ and $R$ are $n$-gons, exactly one vertex of $P$ lies on each side of $R$. Hence no side of $P$ lies on a side of $R$, and therefore the vertices of $R$ are smooth points of the map $f$.

Recall the classification of orientation preserving isometries of the hyperbolic plane (see, e.g., [6]). An isometry is elliptic, parabolic or hyperbolic depending on the number and location of its fixed points: an elliptic isometry has a unique fixed point in the hyperbolic plane (and therefore is a rotation); a parabolic one has a unique fixed point on the circle at infinity; and a hyperbolic one has two fixed points on the circle at infinity, one attracting and one repelling, both of hyperbolic type, i.e., the derivative of the respective circle map at a fixed point is not equal to 1.

Let $T$ be the composition of reflections in the consecutive vertices of $P$. Then $T$ is an orientation preserving isometry, and an $n$-periodic point of $f$ is a fixed point of $T$. Thus $T$ is either parabolic or hyperbolic, and it has 1 or 2 fixed points at infinity, respectively. ∎

Recall that a convex $n$-gon $P$ is called large if $\rho(P) = 1/n$ and the respective isometry $T$ from the proof of Lemma 4 is hyperbolic. Let $x_1, \ldots, x_n \in S^1$ be an $n$-periodic orbit of $f$, and let $\lambda_i$ be the ratio in which a vertex of $P$ divides the segment $x_i x_{i+1}$. The map $T$ is a projective transformation of the plane, and it is smooth in a neighborhood of the closed unit disc. Let $A = D_{x_1}T$ be the derivative of $T$ in its fixed point $x_1$. The eigenvalues of $A$ have the next geometrical interpretation.

**Lemma 5.** *The eigenvalues of $A$ are $\prod_{i=1}^n \lambda_i$ and $\prod_{i=1}^n \lambda_i^2$.*

*Proof.* Consider the field of orthonormal frames $(u, v)$ along $S^1$: the first vector $u$ is the unit tangent vector and $v$ is the inward unit normal vector. Let $x \in S^1$ and $y = S(x)$ where $S$ is the reflection of the hyperbolic plane in a point $O$ — see figure 6. Let $\mu = |Oy|/|Ox|$. We will compute the matrix of the derivative $D_xS$ with respect to the chosen basis. Denote by $\theta$ the angle between the line $xy$ and the circle $S^1$.

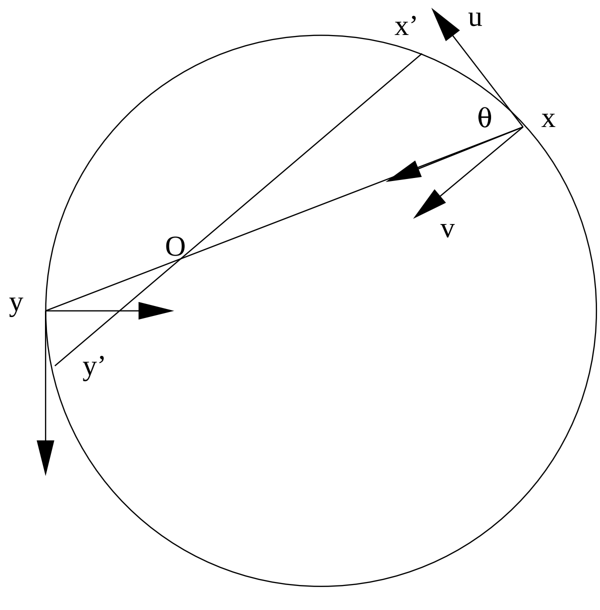


Fig. 6. Computing eigenvalues of the derivative

Clearly, $u$ is an eigenvector of $D_xS$. To find the eigenvalue, let $x' \in S^1$ be a point, close to $x$. The eigenvalue in question equals

$$\lim_{x' \to x} \frac{|y'y|}{|x'x|}.$$

The infinitesimal triangles $xOx'$ and $yOy'$ are similar, therefore the eigenvalue equals $\mu$.

Let $w_x$ be the unit vector at point $x$ in the direction to $y$; then $w_x = (\cos\theta, \sin\theta)$ in the selected frame. Likewise, denote by $w_y$ the unit vector at $y$ in the direction to $x$. Clearly, $D_xS(w_x) = cw_y$. To compute the number $c$, choose a coordinate system on the line $yx$ so that $O$ is the origin. Abusing notation, denote the coordinates of the points $x$, $y$ by the same letters. Let $x'' = x - \varepsilon, y'' = y + \delta$ where $S(x'') = y''$. Then $c = \lim_{\varepsilon\to 0} \delta/\varepsilon$.

One has: $d(y', O) = d(O, x')$, that is,

$$\ln \frac{|x-O||y'-y|}{|x-y'||O-y|} = \ln \frac{|x-x'||O-y|}{|x-O||x'-y|}.$$

Since $O$ is the origin on the line,

$$\frac{\delta x}{y(x-y-\delta)} = \frac{y\varepsilon}{x(x-\varepsilon-y)} \quad \text{or} \quad \frac{y^2(x-y-\delta)}{x^2(x-y-\varepsilon)} = \frac{\delta}{\varepsilon}.$$

It follows that $c = y^2/x^2 = \mu^2$, and a straightforward computation yields the matrix:

$$D_xS = \begin{pmatrix} \mu & -(\mu^2+\mu)\cot\theta \\ 0 & \mu^2 \end{pmatrix}.$$

To complete the proof of the lemma, apply the chain rule taking consecutively $x = x_i, y = x_{i+1}$: the product of upper triangular matrices is upper triangular, and the eigenvalues multiply. ∎

The next lemma provides a justification of the term “large”.

**Lemma 6.** *Let $x_1, \ldots, x_n \in S^1$ be cyclically ordered distinct points on the unit circle. There exists an $\varepsilon > 0$ such that every convex $n$-gon in the hyperbolic plane with vertices $y_i$ within the Euclidean distance $\varepsilon$ from $x_i$ respectively, is large.*

*Proof.* Let $x_i = (\cos\alpha_i, \sin\alpha_i)$, $i = 1, ..., n$. Consider the points

$$x_i(t_i) = (\cos(\alpha_i + t_i), \sin(\alpha_i + t_i)), \ -\varepsilon < t_i < \varepsilon$$

where $\varepsilon$ is sufficiently small. Thus $t_i$ are local coordinates on the circle near points $x_i$. Connecting the consecutive points $x_i(t_i)$, we obtain a convex polygon. Consider the points $y_i$ on the sides of this polygon:

$$y_i = (1-s_i)\ x_i(t_i) + s_i\ x_{i+1}(t_{i+1}), \ 0 < s_i < \varepsilon.$$

By construction, connecting the points $y_i$ gives a polygonal billiard table $P$ with $n$-periodic orbit at infinity and $\rho(P) = 1/n$. By Lemma 5, this periodic orbit is hyperbolic if $\prod s_i/(1-s_i) < 1$, which holds for $s_i$ small enough. Therefore $P$ is large.

Consider the map

$$\mathcal{F} : (s_1, ..., s_n, t_1, ..., t_n) \mapsto (y_1, ..., y_n)$$

from $2n$ dimensional space to $2n$ dimensional space. A direct computation yield the following formula for the Jacobian of $\mathcal{F}$, evaluated at $s_i = 0, t_i = 0$, $i = 1, \ldots, n$:

$$\prod_{i=1}^{n} (1 - \cos(\alpha_i - \alpha_{i+1})).$$

This expression is positive since the points $x_i$ are distinct. It follows that the Jacobian does not vanish for all sufficiently small $s_i, t_i$. Thus $\mathcal{F}$ is a local diffeomorphism, and for every choice of $y_i$ within distance $\varepsilon$ from $x_i$, the respective dual billiard table $P$ is large. ∎

The next lemma gives two further geometrical properties of large polygons.

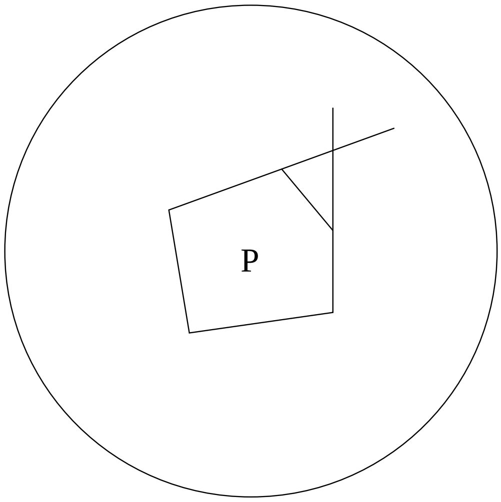


Fig. 7. The sides of large polygon do not intersect

**Lemma 7.** *Let $P$ be a large $n$-gon. Then no convex $k$-gon with $k < n$ can contain $P$. The extensions of the sides of $P$ do not intersect.*

*Proof.* Let $P \subset Q$ and $Q$ be a $k$-gon with $k < n$. Then $\rho(P) = 1/n$, and $\rho(Q) \leqslant \rho(P)$ by Lemma 1. On the other hand, by Proposition 3, $\rho(Q) \geqslant 1/k$. Therefore $1/k \leqslant 1/n$, a contradiction. If the extensions of two sides of $P$ intersect then $P$ is contained inside a convex polygon with fewer sides — see figure 7. This is impossible by the first claim. ■

Now we will prove Theorem 2.

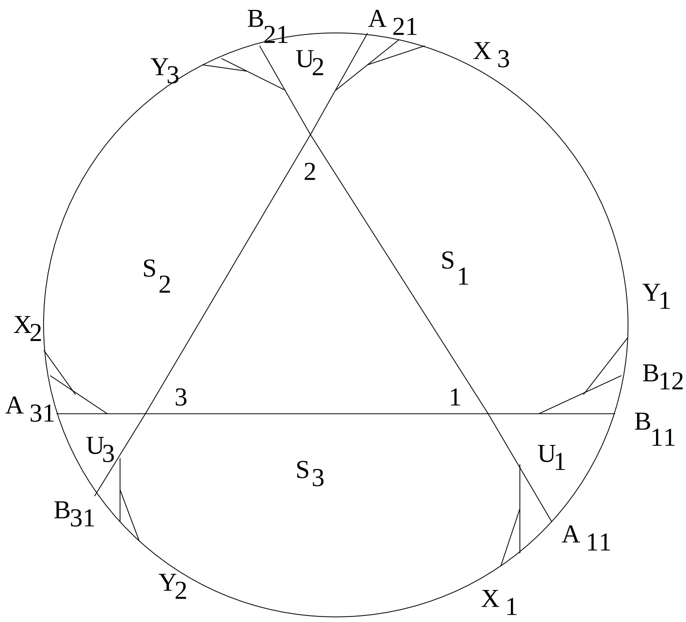


Fig. 8. Proof of Theorem 2

**Proof of Theorem 2**. Label the vertices of a large $n$-gon $P$ counterclockwise by $1, 2, \ldots, n$; as usual, we understand the indices cyclically. Let $l_{i1}$ be the extension of the side $(i+1, i)$ beyond vertex $i$ and set: $A_{i1} = l_{i1} \cap S^1$. As we know, the map $f$ has two $n$-periodic orbits, a repelling and an attracting one. Denote these orbits by $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ — see figure 8.

Consider the consecutive preimages of the points $A_{i1}$ under the map $f$ and set: $A_{ij} = f^{-1}(A_{i+1,j-1})$. Then $A_{ij}$ lies between $A_{i,j-1}$ and $X_i$. Therefore the sequence of points $A_{ij}$, $j = 1, 2, \dots$ has a limit, and this limit is a periodic point of the map $f$. Thus $\lim_{j\to\infty} A_{ij} = X_i$. Since the iterations of points $A$ under the map $f^{-1}$ converge to the points $X$, the periodic orbit $X_i$ is repelling for $f$.

Consider the consecutive preimages of the half-lines $l_{i1}$ under the map $F$ and set: $l_{ij} = F^{-1}(l_{i+1,j-1})$. The set $\cup_{i,j} l_{ij}$ consists of the points at which some forward iteration of $F$ is not defined. Note that each set $\cup_j l_{ij}$ is a tree made of half-lines.

Replacing $f$ and $F$ by $f^{-1}$ and $F^{-1}$, respectively, yields points $B_{ij}$ and half-lines $m_{ij}$ as shown in figure 8. One has: $\lim_{j\to\infty} B_{ij} = Y_i$, and $Y_i$ is an attracting periodic orbit for $f$. Note that the lines $l_{ij}$ and $m_{i'j'}$ are disjoint: this is a generalization of the second property in Lemma 7.

Consider the complement to the singularity set $\Delta = \cup l_{ij} \cup m_{i'j'}$. This complement consists of a countable union of wedges denoted by $U_i, V_{ij}, W_{ij}$ and $n$ domains denoted by $S_i$ — see figure 8 again. Namely, $V_{ij}$ is bounded by $l_{i,j-1}$ and $l_{ij}$, and $W_{ij}$ is bounded by $m_{i,j-1}$ and $m_{ij}$. The dual billiard map permutes these domains as follows:

$$F : U_i \to W_{i+1,2} \to W_{i+2,3} \to W_{i+3,4} \to \dots,$$

$$F^{-1} : U_i \to V_{i-1,2} \to V_{i-2,3} \to V_{i-3,4} \to \dots$$

Therefore the map $F$ consecutively reflects a point $x \in U_i$ in the vertices $i+1, i+2, i+3, \dots$ Let $T_i$ be the composition of $n$ reflections in the vertices $i+1, i+2, \dots, i$. Then $T_i$ is a hyperbolic isometry with the attracting fixed point $Y_i$. It follows that

$$\lim_{k\to\infty} F^{kn}(x) = \lim_{k\to\infty} T_i^k(x) = Y_i.$$

Since each set $V_{ij}$ and $W_{ij}$ is an iterated image or a preimage of some $U_k$ under the map $F$, the forward orbits of points in $\cup V_{ij} \cup W_{i'j'}$ also converge to the periodic orbit $Y$. Finally, consider the sets $S_i$. One has: $F(S_i) = S_{i+1}$. If $x \in S_i$ then $F^{kn}(x) = T_i^k(x) \to Y_i$, so the points in $S_i$ escape to infinity as well. ∎

Next, we prove Theorem 3.

**Proof of Theorem 3**. Let $T$ be the composition of three consecutive reflections in the vertices of a triangle $P$. The map $T$ is an isometry, either an elliptic, or parabolic, or hyperbolic. In any case, $T$ has a fixed point, possibly on the circle at infinity. Connect the images of this fixed point under the consecutive reflections in the vertices of $P$ to obtain a triangle whose sides are bisected by the vertices of $P$. In other words, the dual billiard map with respect to $P$ has a 3-periodic orbit. The triangle $P$ is large if the 3-periodic orbit lies at infinity and is hyperbolic. This corresponds to the case when $T$ is a hyperbolic isometry. Note that this argument may fail for $n$-gons with $n > 3$ — see figure 9.

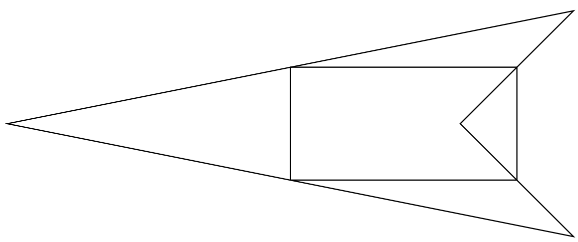

Fig. 9. “Fake” periodic orbit

Recall that, in the upper half-plane model of the hyperbolic plane, orientation preserving isometries are represented by fractional-linear transformations with real coefficients; these fractional-linear transformations are considered as elements of the group $PSL(2,\mathbf{R})$. The reflection at point $p+iq, q>0$ is given by the formula:

$$z \mapsto \frac{pz-p^2-q^2}{z-p}.$$

Recall also that an isometry is hyperbolic if and only if the absolute value of the trace of the respective element in $PSL(2,\mathbf{R})$ is greater than 2.

Let $p_1+iq_1, p_2+iq_2$ and $p_3+iq_3$ be the three vertices of the triangle. A straightforward calculation of the product of three matrices yields the following formula:

$$\operatorname{Tr}(T) = \frac{\sum_{i=1}^{3} p_i \ (p_{i+1}^2 + q_{i+1}^2 - p_{i-1}^2 - q_{i-1}^2)}{q_1 q_2 q_3}.$$

To interpret this expression geometrically, we may assume that the first two vertices lie on the vertical axis, that is, $p_1 = p_2 = 0$ and $q_1 > q_2$. Then

$$\operatorname{Tr}(T) = \frac{p_3 \ (q_1^2 - q_2^2)}{q_1 q_2 q_3}.$$

Let $a$ be the length of the side of the triangle that lies on the vertical axis and $h$ the length of the altitude dropped on this side. We claim that

$$\frac{q_1^2 - q_2^2}{q_1 q_2} = 2\sinh a, \quad \frac{p_3}{q_3} = \sinh h. \tag{2.1}$$

Assuming this claims, the triangle is large if and only if

$$2 < \operatorname{Tr}(T) = 2\sinh a \ \sinh h,$$

which implies one of the statements of Theorem 3. The other formulas for $H$ in the formulation of this theorem are contained in [6], exercises 20–28, pp. 432-434.

It remains to prove (2.1). Recall that the metric in the upper half-plane model is given by the formula:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Therefore

$$a = \int_{q_2}^{q_1} \frac{dy}{y} = \ln q_1 - \ln q_2,$$

and hence

$$2\sinh a = \frac{q_1^2 - q_2^2}{q_1 q_2}.$$

The perpendicular from point $p_3 + iq_3$ on the vertical axis is represented by an arc of the circle through $p_3 + iq_3$ and centered at the origin. This arc is parameterized as

$$\sqrt{p_3^2 + q_3^2} \ (\cos t, \sin t), \ t \in [\arctan{(q_3/p_3)}, \pi/2].$$

It remains to integrate the 1-form $ds$ along this curve which yields

$$h = \ln\tan\Big(\frac{\arctan(q_3/p_3)}{2}\Big),$$

and therefore $\sinh h = p_3/q_3$. ■

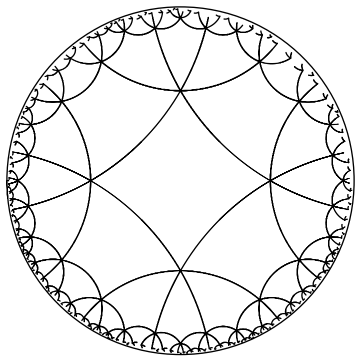

Fig. 10. Tiling by $(4, 6)$-gons

## 3. Regular right angled $n$-gons

Let $n$ and $k$ satisfy $1/n + 1/k < 1/2$. An $(n, k)$-gon is a regular $n$-gon with the interior angle $2\pi/k$; such a polygon tiles the hyperbolic plane so that $k$ tiles meet at a vertex — see figure 10. If $k$ is even then this tiling is invariant under reflections in the vertices of the $(n, k)$-gon. Therefore the respective dual billiard map sends tiles to tiles, and one consider the induced action on the set of tiles which we also denote by $F$.

The next result shows that most $(n, k)$-gons are large, in which cases the dynamics of the dual billiard map is not interesting.

**Lemma 1.** *The $(n, k)$-gon is not large only in the following cases: $(i)$ $k = 3, n \geqslant 7$, $(ii)$ $k = 4, n \geqslant 5$ or $(iii)$ $k = 5, n = 4$.*

*Proof.* Let $R$ be the regular $n$-gon inscribed into the unit circle, and $Q$ is the regular $n$-gon whose vertices bisect, in the Euclidean sense, the sides of $R$ — see figure 11. Then $Q$ is the smallest regular $n$-gon such that the map $f$ has the rotation number $1/n$. Let $P$ be the $(n, k)$-gon. Then $P$ is large if and only if it contains $Q$. Let $r$ be the distance from the center of the unit circle $O$ to a vertex of $Q$ and let $d$ be the distance from the center of $P$ to its vertex. Then $P$ is large if and only if $d > r$.

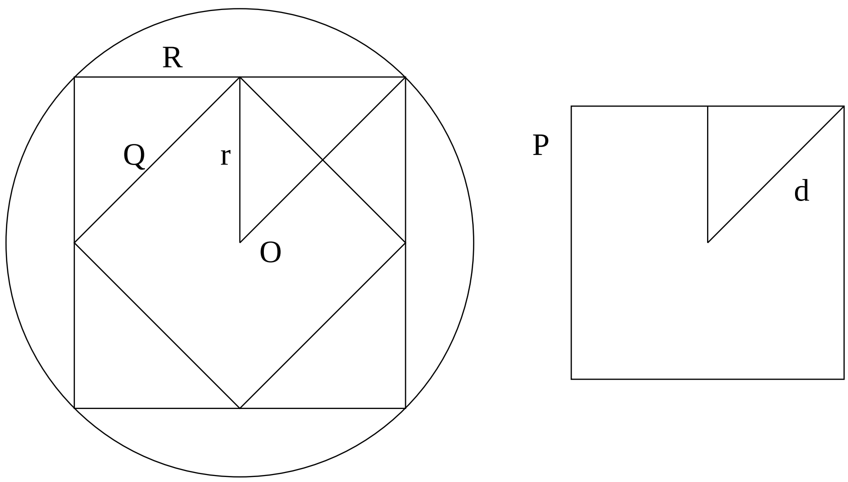


Fig. 11. When $(n, k)$-gon is large

To compute $r$ connect $O$ with a vertex of $Q$ and with an adjacent vertex of $R$. One obtains a right triangle with one zero angle at infinity and the other acute angle $\pi/n$. By hyperbolic Cosine Rule, $\cosh r = 1/\sin(\pi/n)$. Similarly, connect the center of $P$ with its vertex and with the midpoint of an adjacent side. One obtains a right triangle with acute angles $\pi/k$ and $\pi/n$ and the hypotenuse $d$.

By hyperbolic Cosine Rule, $\cosh d = \cot(\pi/k)\cot(\pi/n)$. It follows that $P$ is large if and only if $\cosh r < \cosh d$, or equivalently, $\tan(\pi/k) < \cos(\pi/n)$.

This inequality obviously does not hold for $k = 3, 4$. The left hand side decreases as $k$ increases, and the right hand side increases as $n$ increases. One concludes that the only remaining case when $\tan(\pi/k) \geqslant \cos(\pi/n)$ is $k = 5, n = 4$. ■

In this section we study the case when the dual billiard table $P$ is an $(n, 4)$-gon, $n \geqslant 5$. Let $Q$ be a tile in the tiling of the hyperbolic plane determined by $P$. Define *rank*, rk $Q$, as the length of the shortest chain of tiles from $P$ to $Q$ such that the adjacent tiles share a side. For example, $P$ has $n$ neighbors of rank 1; the number of rank 2 tiles is $n(n-2)$. The ranks of tiles that share a side differ by 1. The tiles of a fixed rank make a "necklace" around $P$ with the adjacent tiles sharing a vertex — see figure 12 for $n = 5$. We order the tiles of a fixed rank cyclically.

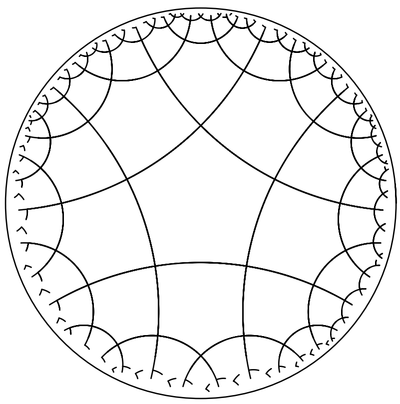

Fig. 12. Tiling by $(5, 4)$ polygons

**Theorem 4.** *The dual billiard map $F$ preserves the rank of a tile, and every orbit of $F$ is periodic. The set of tiles of rank $k$ consists of*

$$q_k = n\frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}$$

*elements where*

$$\lambda_{1,2} = \frac{n - 2 \pm \sqrt{n(n-4)}}{2}$$

*are the roots of the equation $\lambda^2 - (n-2)\lambda + 1 = 0$. The action of $F$ on the set of tiles of rank $k$ is a transitive cyclic permutation $i \mapsto i + p_k$ where*

$$p_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} + \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2}.$$

*The rotation number of the dual billiard map at infinity is given by the formula:*

$$\rho(f) = \lim_{k\to\infty}\frac{p_k}{q_k} = \frac{n - \sqrt{n(n-4)}}{2n}.$$

*Proof.* Let us prove that the rank of a tile is preserved by the dual billiard map. This is clearly so for rank 1 tiles. Assume this is true for ranks not greater than $k-1$. Let $Q$ be a tile of rank $k$. Then $Q$ has an adjacent tile $R$ of rank $k-1$. If the common side of $Q$ and $R$ does not lie on the extension of a side of $P$ then the tiles $F(Q)$ and $F(R)$ are also adjacent. Since rk $F(R) = k-1$ one has rk $F(Q) \leqslant k$.

Consider the case when the common side of $Q$ and $R$ lies on the extension of a side of $P$. Then $Q$ has another adjacent tile, $R'$, of rank $k-1$, and their common side does not belong to the discontinuity set of $F$ — figure 13. Arguing as before, rk $F(Q) \leqslant k$. By the induction assumption, $F$ preserves the set of tiles of rank less than $k$. Therefore $F$ also preserves the set of rank $k$ tiles.

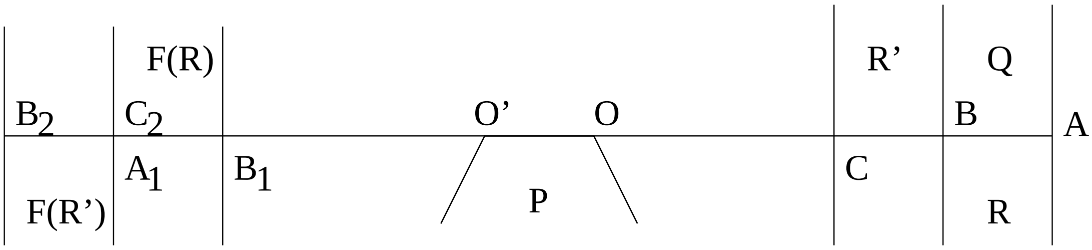


Fig. 13. Action on the set of tiles of a fixed rank

Thus every tile $Q$ is periodic under the map $F$; let $m$ be the period. Since the isometry $F^m$ preserves $Q$, this map is a rotation about the center of $Q$ through an angle which is a multiple of $2\pi/n$. Therefore every point of $Q$ is $mn$-periodic.

Let us prove that $F$ permutes the tiles of a fixed rank cyclically. Let $R$ and $R'$ be two consecutive tiles in the necklace of rank $k$ tiles. If $R$ and $R'$ are not separated by the extension of a side of $P$ then $R$ and $R'$ reflect in the same vertex of $P$, and the tiles $F(R)$ and $F(R')$ are also adjacent rank $k$ tiles.

Consider the case when $R$ and $R'$ are separated by the extension of a side $O'O$ of $P$ – see figure 13. Then $R$ reflects in $O$ and $R'$ in $O'$. We claim that the tiles $F(R)$ and $F(R')$ are still adjacent and their cyclic order does not change. Let $A_1B_1$ be the reflection of $AB$ in $O$ and $B_2C_2$ be the reflection of $BC$ in $O'$. We need to show that $C_2 = A_1$, or equivalently, $CO' = O'A_1$. Indeed, $AB = BC = OO' = A_1B_1 = B_2C_2$ and

$$O'A_1 = OA_1 - OO' = OA - OO' = O'B - OO' = O'C,$$

as claimed.

Let us find the number of rank $k$ tiles $q_k$. Let $l, l', l''$ be three consecutive sides of $P$. Consider the region between $l$ and $l''$ and call it the region of the first kind; let $t_k$ the number of rank $k$ tiles in this region. Consider also the wedge bounded by $l$ and $l'$ and call it the region of the second kind; let $s_k$ the number of rank $k$ tiles in this region — see figure 14. Since $P$ enjoys $n$-fold rotation symmetry, one has: $q_k = n(t_k + s_k)$.

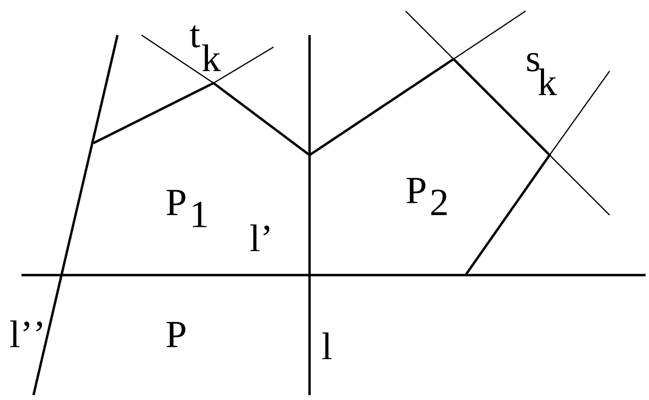


Fig. 14. Counting tiles of a fixed rank

We claim that the numbers $t_k$ and $s_k$ satisfy the following recurrence relations:

$$t_k = (n-3)t_{k-1} + (n-4)s_{k-1}, \quad s_k = (n-2)t_{k-2} + (n-3)s_{k-2}. \tag{3.1}$$

To prove this claim, consider a tile $Q$ of rank $k$ in the region of the first kind on figure 14. Then length of the shortest chain of tiles connecting $Q$ and $P_1$ is $k-1$. In other words, if $P_1$ is chosen as the tile of reference (instead of $P$) then rk $Q$ would be $k-1$. The region of the first kind with respect to $P$ consists of $n-3$ regions of the first kind and $n-4$ regions of the second kind with respect to $P_1$ — see figure 14. This implies the first relation (3.1). One argues similarly about the region of the second kind and the tile $P_2$ that has rank 2, and this proves the second relation (3.1).

To solve the recurrence relations, substitute $t$ from the first formula to the second to obtain

$$s_{k+3} - (n-3)s_{k+2} - (n-3)s_{k+1} + s_k = 0. \tag{3.2}$$

Clearly, one also has the initial conditions $s_1 = 0, s_2 = 1, s_3 = n-2$. The characteristic polynomial of (3.2) is $(\lambda+1)(\lambda^2 - (n-2)\lambda + 1)$. Solving the recurrence (3.2) one finds:

$$s_k = \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2}, \quad t_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} - \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2}. \tag{3.3}$$

The formula for $q_k$ follows.

To find $p_k$ consider figure 13 again. The map $F$ takes the tile $R$ across one region of the first kind and two regions of the second kind. Therefore $p_k = t_k + 2s_k$, and (3.3) implies the formula for $p_k$.

The cyclic permutation of the rank $k$ tiles by $F$ is a discrete approximation of the circle map $f$. Hence $\rho(f) = \lim_{k\to\infty} p_k/q_k$. Since $\lambda_1 > 1$ and $0 < \lambda_2 < 1$ this limit equals $(1+\lambda_1^{-1})/n$, as claimed.

Finally, we prove that $F$ acts transitively on the set of rank $k$ tiles, or equivalently, that $p_k$ and $q_k$ are coprime. Let $r_k = q_k/n$. Then $r_k = (n-2)r_{k-1} - r_{k-2}$. It also follows from the formulas for $q_k$ and $p_k$ that $p_k = r_k + r_{k-1}$. One has:

$$(p_k, r_k) = (r_k + r_{k-1}, r_k) = (r_{k-1}, r_k) = (r_{k-1}, (n-2)r_{k-1} - r_{k-2}) = (r_{k-2}, r_{k-1}).$$

and one continues this way until one reaches $r_1 = 1$. Thus $(p_k, r_k) = 1$.

Since $q_k = nr_k$, it remains to show that $(p_k, n) = 1$. One has:

$$p_k = (n-2)p_{k-1} - p_{k-2} \quad \text{and} \quad p_1 = 1,\ p_2 = n-1.$$

It follows that $p_k = (-1)^{k+1} \bmod n$, and we are done. ■

**Remark 2.** The numbers $r_k = q_k/n$ have the following relation to the irrational number $\lambda_1$. Since $\lambda^2 = (n-2)\lambda - 1$, one has $\lambda = n - 2 - 1/\lambda$, and therefore

$$\lambda_1 = n - 2 - \cfrac{1}{n - 2 - \cfrac{1}{n - 2 - \cfrac{1}{n - 2 - \dots}}} \tag{3.4}$$

We claim that $r_{i+1}/r_i$ equals the fraction obtained by truncating the continued fraction (3.4) at place $i$. Indeed, this is true for $i = 1$. It was mentioned in the proof of Theorem 4 that $r_k = (n-2)r_{k-1} - r_{k-2}$, or equivalently,

$$n - 2 - \frac{r_{k-2}}{r_{k-1}} = \frac{r_k}{r_{k-1}}.$$

This is how two consecutive truncations of (3.4) are related, and our claim follows by induction.

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