

**COMPLEXITY OF PIECEWISE CONVEX TRANSFORMATIONS
IN TWO DIMENSIONS, WITH APPLICATIONS TO
POLYGONAL BILLIARDS ON SURFACES OF CONSTANT
CURVATURE**

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To A. A. Kirillov for his 70th birthday

ABSTRACT. We introduce piecewise convex transformations, and develop geometric tools to study their complexity. We apply the results to the complexity of polygonal inner and outer billiards on surfaces of constant curvature.

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INTRODUCTION

Transformations that arise in geometric dynamics often fit into the following scheme. The phase space of the transformation, $T: X \rightarrow X$, has a finite decomposition $\mathcal{P}: X = X_a \cup X_b \cup \dots$ into “nice” subsets with respect to a geometric structure on X . (For instance, X_a, X_b, \dots are convex.) The restrictions $T: X_a \rightarrow X$, $T: X_b \rightarrow X, \dots$ preserve that structure. The interiors of X_a, X_b, \dots are disjoint, and T is discontinuous (or not defined) on the union of their boundaries, $\partial\mathcal{P} = \partial X_a \cup \partial X_b \cup \dots$. Set $\mathcal{P}_1 = \mathcal{P}$. The sets $T^{-1}(X_a) \cap X_a, T^{-1}(X_a) \cap X_b, \dots$ form the decomposition \mathcal{P}_2 , which is a refinement of \mathcal{P}_1 ; it provides a defining partition for the transformation T^2 . Iterating this process, we obtain a tower of decompositions $\mathcal{P}_n, n \geq 1$, where \mathcal{P}_n plays for T^n the same role that \mathcal{P} played for T .

Let $\mathcal{A} = \{a, b, \dots\}$ be the alphabet labeling the atoms of \mathcal{P} . A phase point $x \in X$ is regular if every point of the orbit x, Tx, T^2x, \dots belongs to the interior of an atom of \mathcal{P} . Let $x \in X_a, Tx \in X_b, \dots$. The word $\sigma(x) = ab\dots$ is the *code* of x . Let $\Sigma(n)$ be the set of words of length n obtained this way. The function $f(n) = |\Sigma(n)|$ is the *complexity* associated with the triple X, \mathcal{P}, T . Its behavior as $n \rightarrow \infty$ is an important characteristic of the dynamical system in question. We will develop a geometric approach to complexity.

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Several classes of transformations (e.g., piecewise isometries, piecewise affine mappings, etc.) fit into the scheme above. The following examples have directly motivated our study.

Example A. Let $P \subset \mathbb{R}^2$ be a polygon with sides a, b, \dots , and let X be the phase space of the *billiard map* T_{bil} in P . Its elements are the *oriented segments*. Let X_a, X_b, \dots be the sets of segments that begin in a, b, \dots . The decomposition $\mathcal{P}: X = X_a \cup X_b \cup \dots$ yields the coding of billiard orbits by the sides that they hit [16]. Basic questions about its complexity are open [10].

Example B. Let $P \subset \mathbb{R}^2$ be a convex polygon with corners a, b, \dots . The complement $X = \mathbb{R}^2 \setminus P$ is the phase space of the *outer billiard* T^{out} about P . See Definition 7 below. (It is also called the *dual billiard* [16].) Let $X_a, X_b, \dots \subset \mathbb{R}^2 \setminus P$ be the conical regions bounded by its singularities. In X_a, X_b, \dots the mapping T^{out} is the symmetry about a, b, \dots . The decomposition $\mathcal{P}: X = X_a \cup X_b \cup \dots$ yields the coding of outer billiard orbits by the corners that they hit.

We will study the complexity of (2-dimensional) *piecewise convex transformations*. This is a wide class of geometric dynamical systems; in particular, it contains the examples above. Our setting is as follows.

A *chord space* is a topological space such that for any pair of distinct points x_0, x_1 there is a unique *chord* $[x_0, x_1]$ joining them. See examples of chord spaces in Section 1. A *convex cell complex* is a cell complex whose closed cells are chord spaces, and the chord structures agree on the intersections. Let $\dim X = 2$, and let $\mathcal{P}: X = \overline{X}_a \cup \overline{X}_b \cup \dots$ be the decomposition by the closed 2-cells. Suppose that there are homeomorphisms $T_a: \overline{X}_a \rightarrow X, T_b: \overline{X}_b \rightarrow X, \dots$ such that for any chord $\gamma \subset X_a$ and any 2-cell $X_b \subset X$ the curve $T_a(\gamma) \cap \overline{X}_b$ is a chord. We will say that (X, \mathcal{P}, T) is a *piecewise convex transformation*.

We will use these transformations to study the inner and the outer polygonal billiards on surfaces of constant curvature, \varkappa . Using the natural normalization, we will speak of the elliptic ($\varkappa = 1$), the hyperbolic ($\varkappa = -1$), and the parabolic ($\varkappa = 0$) cases. One of our goals is to develop a uniform approach to these dynamical systems.

Let $P \subset M$ be a geodesic polygon on a surface of constant curvature. Since the outer billiard about P is a piecewise isometry, it directly fits into the framework of piecewise convex transformations. In Section 2 we put the inner billiard into this framework. In order to do this, we modify the definition of the *billiard phase space*. Our phase space, $X = X(P)$, is the quotient of the set of billiard segments in P by an equivalence relation. See Definition 5. Endowed with the quotient topology and the natural chord structure, X is a cell complex; it is also a (finite, branched) covering of the space, $\mathcal{L} = \mathcal{L}(P)$, of rays intersecting P . See Theorem 1. In Section 2 we develop a dictionary that translates the statements of Section 1 into the language of billiard orbits. See Proposition 4.

In Section 3 we investigate the inner billiard complexity. First, we establish the background by considering arbitrary polygons, and any constant curvature. Then we study each of the three cases separately. Below is a sample of our results.

Let $\varkappa = 0$. The side complexity of billiard orbits in any rational polygon grows at most cubically. See Theorem 2.

Let $\varkappa = 1$. The side complexity of billiard orbits is subexponential. See Theorem 3.

Let $\varkappa = -1$. The side complexity of billiard orbits grows exponentially; the exponent in question is the topological entropy h_{top} of the billiard map. See Theorem 4.

In Section 4 we investigate complexity of the polygonal outer billiard on surfaces of constant curvature. Here are some of our results.

Let $\varkappa = 0$. For an arbitrary (resp. rational) polygon the complexity has polynomial (resp. quadratic) bounds from above and from below. See Theorem 5 and Theorem 6.

Let $\varkappa = 1$. The complexity is subexponential. See Theorem 7.

Let $\varkappa = -1$. For an arbitrary polygon, the complexity has a sharp linear lower bound; for *large polygons* the complexity grows linearly. See Theorem 8.

Remark 1. The idea to relate the side complexity with the geometry and combinatorics of billiard orbits goes back to [5]. (The discussion there is restricted to convex, euclidean polygons.) Claim 2 of our Theorem 2 is contained in [5]. The structure of billiard singularities in convex polygons is much simpler than in general polygons: Certain singularities that we had to account for in the proof of Lemma 2 do not occur for convex polygons. This circumstance allowed [5] to obtain an analog of our Lemma 4 directly from Euler's identity. See the proof of Lemma 3.1 in [5]. The paper N. Bedaride, *Billiard complexity in rational polyhedra*, Regular & Chaotic Dynamics 8 (2003) contains an attempt to adapt Lemma 3.1 to nonconvex polygons.

The class of piecewise convex transformations put forward here provides a natural framework to study the coding complexity. This class is general enough to contain the billiard in any polygon on a surface of constant curvature. However, putting the general billiard into the framework of piecewise convex transformations is far from straightforward. See the discussion in Section 2.

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1. PIECEWISE CONVEX TRANSFORMATIONS: GEOMETRY AND COMBINATORICS

1.1. Chord spaces and convex cell complexes. A topological space, X , is a *chord space* if for any pair of distinct points $x_0, x_1 \in X$ there is a unique *chord* $[x_0, x_1] \subset X$ joining them. We require that the chords satisfy the standard properties [3]. A *mapping of chord spaces* is a continuous mapping $f: X \rightarrow Y$ that sends

chords to chords. A subset $Y \subset X$ of a chord space is chord-convex¹ if for any $y_0, y_1 \in Y$ we have $[y_0, y_1] \subset Y$. Then Y itself is a chord space. The *convex hull*, $\text{hull}(Y) \subset X$, of any subset $Y \subset X$ is a chord space.

Example 1. (i) Let X be a *Hadamard manifold*, i. e., a simply connected, complete riemannian manifold of nonpositive sectional curvature. Set $[x, y]$ be the unique connecting geodesic. Then X is a chord space.

(ii) For a pair $x, y \in \mathbb{R}P^n$ of distinct points in the real projective space let $l = l(x, y) \subset \mathbb{R}P^n$ be the projective line containing them. Let $X \subset \mathbb{R}P^n$ be a convex subset, disjoint from a hyperplane. Setting $[x, y] = l(x, y) \cap X$ makes X a chord space.

We will encounter situations where the chord joining $x_0, x_1 \in X$ does not always exist. For example, this happens if X is a nonconvex subset of a chord space. We will say that X is a *space with a chord structure*. Mappings of spaces with chord structures are defined the same way as the mappings of chord spaces.

Example 2. Let \mathcal{R} be the space of *euclidean rays* (i. e., oriented straight lines) in \mathbb{R}^2 . Endowed with the natural topology, \mathcal{R} is the infinite cylinder [15].

Two rays l_0, l_1 are parallel (resp. *antiparallel*) if the corresponding lines are parallel, and their directions are the same (resp. opposite). Let $l_0, l_1 \in \mathcal{R}$ be distinct rays. If they are not parallel or antiparallel, then l_0, l_1 intersect at a point, say $o \in \mathbb{R}^2$, forming a cone, $C \subset \mathbb{R}^2$, with the apex at o . The chord $[l_0, l_1]$ consists of the rays passing through o and contained in C . If l_0, l_1 are parallel, then C becomes a strip, and $[l_0, l_1]$ is defined analogously. If l_0, l_1 are antiparallel, then $[l_0, l_1]$ is not defined. This is a chord structure on \mathcal{R} . Geometrically, $[l_0, l_1]$ is the *pencil of rays* interpolating between l_0, l_1 .

Example 3. Let $\mathcal{R}_{\mathbb{H}} \subset \mathcal{R}$ be the set of rays intersecting the unit disc. We endow $\mathcal{R}_{\mathbb{H}}$ with the induced chord structure. Let $l_0, l_1 \in \mathcal{R}_{\mathbb{H}}$ be distinct rays. By definition, the chord $[l_0, l_1]_{\mathbb{H}}$ exists iff $[l_0, l_1]$ exists and $[l_0, l_1] \subset \mathcal{R}_{\mathbb{H}}$. Let AB, CD be the chords of the unit disc corresponding to l_0, l_1 respectively. Then $[l_0, l_1]_{\mathbb{H}}$ does not exist iff the points A, B, C, D of the unit circle are in a cyclic order. See Fig. 1.

For simplicity of exposition, we will restrict our considerations from here on to two dimensions. A *cell complex* is a topological space, X , endowed with a *cell decomposition*. By our assumption, a cell complex has zero-cells (vertices), one-cells (edges), and two-cells (faces). Each cell $C \subset X$ is homeomorphic to the open disc of the same dimension. The boundary $\partial C \subset X$ is homeomorphic to the sphere; it is a finite union of cells of smaller dimension.

Definition 1. Let X be a cell complex. Suppose that each closed cell $\overline{C} \subset X$ is a chord space, and that the chord structures agree on the intersections $\overline{C'} \cap \overline{C''}$. Then X is a *convex cell complex*.

Let X be a convex cell complex, and let $\Gamma \subset X$ be the union of closures of one-cells. Then Γ is a graph, and its edges are chords; we will say that Γ is a *convex (chord) graph*. We will also speak of spaces with *convex graphs*, and use the notation (X, Γ) .

¹ We will simply say *convex* in what follows.

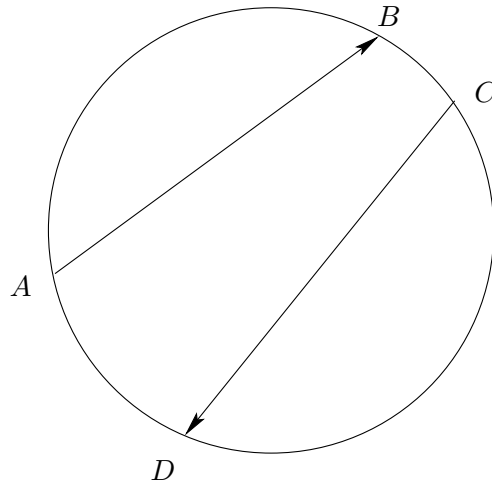


FIGURE 1. Elements of $\mathcal{R}_{\mathbb{H}}$ that cannot be joined by a chord

Definition 2. Let (X, Γ) and (Y, Δ) be convex cell complexes. Let $\varphi: X \rightarrow Y$ be a continuous mapping. Suppose that for any cell $D \subset Y$ the preimage $f^{-1}(D) = C_1 \cup \dots \cup C_n$ is a (nonempty) disjoint union of cells, and that the maps $\varphi: C_i \rightarrow D$ are isomorphisms of chord spaces. Then $\varphi: X \rightarrow Y$ is a (branched) covering of convex cell complexes. The maximal value of $n = n(D)$ is the degree of the covering.

Let X be a topological space, and let Y be a cell complex. A continuous, surjective mapping $f: X \rightarrow Y$ is a (branched) topological covering if for any cell $D \subset Y$ we have $f^{-1}(D) = C_1 \cup \dots \cup C_n$, a disjoint union, and the restrictions $f: C_i \rightarrow D$ are homeomorphisms.

Lemma 1. Let X (resp. (Y, Δ)) be a topological space (resp. a convex cell complex), and let $f: X \rightarrow Y$ be a (branched) topological covering. Then there is a unique convex cell complex (X, Γ) such that $f: (X, \Gamma) \rightarrow (Y, \Delta)$ is a (branched) covering of convex cell complexes.

Proof. The representations $f^{-1}(D) = C_1 \cup \dots \cup C_n$, where $n = n(D)$ and D runs through the cells of Y , define the cells of X , and the unique chord structures on them such that $f|_{C_i}: C_i \rightarrow D$ are isomorphisms of chord spaces. Setting $\Gamma = f^{-1}(\Delta)$, we obtain the claim. \square

We will say that the convex cell complex (X, Γ) of Lemma 1 is induced by the mapping $f: X \rightarrow Y$.

1.2. Piecewise convex transformations. We will now define a class of dynamical systems that will provide a common framework for several kinds of geometric transformations with singularities.

Definition 3. Let (X, Γ) be a convex cell complex. Suppose that for each 2-cell $F \subset X$ we have a homeomorphism $T_F: \bar{F} \rightarrow X$ such that for any chord $\gamma \subset F$ and

any 2-cell $G \subset X$ the curve $T_F(\gamma) \cap \overline{G}$ is a chord. Then (X, Γ, T) is a *piecewise convex self-mapping*.

Let (X, Γ, T) and (X, Δ, S) be piecewise convex self-mappings. They are *mutually inverse* if $T \circ S(x) = S \circ T(x) = x$ for all $x \in X \setminus (\Gamma \cup \Delta)$.

Definition 4. A *piecewise convex transformation* (X, Γ, T) is an invertible piecewise convex self-mapping. We will use the notation $(X, \Gamma, T)^{-1} = (X, \Gamma_{-1}, T^{-1})$.

If (X, Γ) is a convex cell complex, we denote by $\mathcal{P}(\Gamma)$ the associated representation of X as the union of closed faces of Γ ; we will refer to it as a *convex partition*.² If $\Gamma', \Gamma'' \subset X$ are convex graphs, their *join* $\Gamma' \vee \Gamma''$ is also a convex graph, and

$$\mathcal{P}(\Gamma' \vee \Gamma'') = \mathcal{P}(\Gamma') \vee \mathcal{P}(\Gamma''). \tag{1}$$

We outline a proof of this identity. A face, F , of $\Gamma' \vee \Gamma''$ is a connected component of $X \setminus (\Gamma' \cup \Gamma'')$. For $x, y \in F$ the chord $[x, y]$ avoids Γ', Γ'' . Thus, $[x, y]$ belongs to unique faces F' of Γ' and F'' of Γ'' , i. e., $F \subset F' \cap F''$. The converse also holds. Hence $F = F' \cap F''$, implying equation (1).

Let (X, Γ, T) be a piecewise convex transformation. Setting $\Gamma_1 = \Gamma$ and $\Gamma_{n+1} = \Gamma_n \vee T^{-1}(\Gamma_n)$, we inductively define an increasing tower $\Gamma_k, k \geq 1$, of convex graphs on X . By construction, Γ_k is the singular set of T^k , and the piecewise convex transformation (X, Γ_k, T^k) is the k th iteration of T . Let $\mathcal{P}_k = \mathcal{P}(\Gamma_k)$. The set $S_\infty = \bigcup_{k=1}^\infty \Gamma_k$ is a countable (at most) union of chords. The complement $X_\infty = X \setminus S_\infty$ is the set of points $x \in X$ such that x, Tx, T^2x, \dots belong to open faces of Γ . We will refer to them as *regular points*. Iterating (X, Γ_{-1}, T^{-1}) , we obtain the sequence of convex graphs $\Gamma_{-k}, k \geq 1$, and the piecewise convex transformations (X, Γ_{-k}, T^{-k}) , inverse to (X, Γ_k, T^k) .

Let $\mathcal{A} = \{a, b, \dots\}, p = |\mathcal{A}|$, be a set labeling the faces of Γ , and let \mathcal{L} be the *full shift space* on the *alphabet* \mathcal{A} . Assigning to a point $x \in X_\infty$ the sequence of labels of the faces of Γ containing x, Tx, T^2x, \dots we obtain the *coding map* $\sigma: X_\infty \rightarrow \mathcal{L}$. Set $\Sigma = \sigma(X_\infty)$, and let $\Sigma(n)$ be the set of words of length n that occur in Σ . The function $f(n) = |\Sigma(n)|$ is the *complexity* of (X, Γ, T) . The proposition below summarizes the discussion.

Proposition 1. *Let (X, Γ, T) be a piecewise convex transformation. Then there is a sequence $\Gamma_k, k \geq 1$, of convex graphs in X such that the iterations of T correspond to piecewise convex transformations (X, Γ_k, T^k) . There is a natural bijection between $\Sigma(n)$ and the set of faces of the convex graph Γ_n ; the complexity of (X, Γ, T) satisfies $f(n) = |\mathcal{P}(\Gamma_n)|$.*

1.3. Joins of convex graphs: A combinatorial formula. Let X be a compact space with a chord structure. Let $\Gamma', \Gamma'' \subset X$ be convex chord graphs. Set $\Gamma = \Gamma' \vee \Gamma''$.

Denote by $F', F'', F, E', E'', E, V', V'', V$ the sets of faces, edges and vertices of $\Gamma', \Gamma'', \Gamma$ respectively. Arbitrary edges $e' \in E', e'' \in E''$ intersect transversally, or overlap. The latter can occur in four ways. See Fig. 2. Denote by $c(\Gamma', \Gamma'')$ the number of overlappings.

²It is not a set-theoretic partition of X ; convex partitions provide an alternative approach to our material [11].

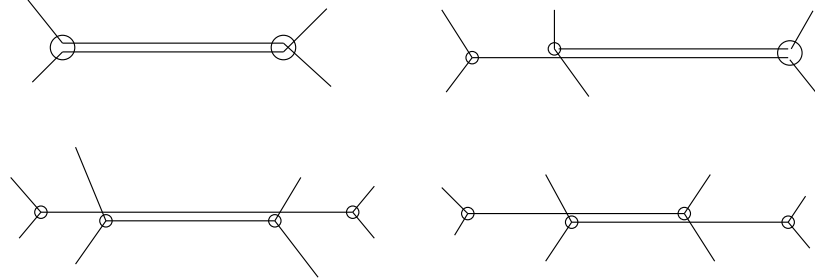


FIGURE 2. Overlapping of edges of two chord graphs

Lemma 2. *Let $\Gamma', \Gamma'' \subset X$ be convex chord graphs, and let $\chi = \chi(X)$ be the Euler number. Let V'_d, V''_d be the sets of vertices of Γ', Γ'' respectively, disjoint from the other graph. Set $V_{\text{ess}} = V \setminus (V'_d \cup V''_d)$. Then*

$$|F| - |F'| - |F''| + \chi = |V_{\text{ess}}| - c(\Gamma', \Gamma''). \tag{2}$$

Proof. Any convex chord graph $A \subset X$ satisfies $|F(A)| - |E(A)| + |V(A)| = \chi$. Using this identity, we obtain

$$|F| - |F'| - |F''| + \chi = (|E| - |E'| - |E''|) + (|V'| + |V''| - |V|). \tag{3}$$

Denote by e'_i, e''_j the edges of Γ', Γ'' respectively. Let a'_i, a''_j be the number of vertices of Γ'', Γ' inside the edges e'_i, e''_j respectively. Let b'_i, b''_j be the number of times that e'_i, e''_j transversally intersects an edge of Γ'', Γ' respectively. Then e'_i, e''_j contribute $a'_i + b'_i + 1, a''_j + b''_j + 1$ edges to E respectively. Taking the overlapping into account, we obtain

$$|E| = \sum_i a'_i + \sum_i b'_i + |E'| + \sum_j a''_j + \sum_j b''_j + |E''| - c(\Gamma', \Gamma''). \tag{4}$$

Let V_c be the set of common vertices of Γ', Γ'' . Let V'_e, V''_e be the sets of vertices of Γ', Γ'' respectively that belong to edges of the other graph. Let $V_n = V \setminus (V' \cup V'')$ be the set of “new” vertices of Γ . Then

$$\begin{aligned} |V'| &= |V'_e| + |V'_d| + |V_c|, & |V''| &= |V''_e| + |V''_d| + |V_c|, \\ |V| &= |V'| + |V''| - |V_c| + |V_n|. \end{aligned} \tag{5}$$

Besides

$$\sum_i a'_i = |V''_e|, \quad \sum_j a''_j = |V'_e|, \quad \sum_i b'_i = \sum_j b''_j = |V_n|. \tag{6}$$

From equations (4-6), we have

$$|E| - |E'| - |E''| = |V'_e| + |V''_e| + 2|V_n| - c(\Gamma', \Gamma'').$$

Substituting this into equation (3), and using equation (5), we obtain the claim. \square

The corollary below concerns a few special cases of Lemma 2.

Corollary 1. *Suppose, in addition to the assumptions of Lemma 2, that $\chi = 1$, and that the edges of graphs Γ', Γ'' do not overlap. Then*

$$|F| - |F'| - |F''| + 1 = |V_{\text{ess}}| \tag{7}$$

and

$$|V_n| \leq |F| - |F'| - |F''| + 1 \leq |V|. \tag{8}$$

If $V'_d = V''_d = \emptyset$ then

$$|F| - |F'| - |F''| + 1 = |V|. \tag{9}$$

Proof. The claims are immediate from Lemma 2 and equation (5). \square

1.4. Complexity of piecewise convex transformations: a geometric formula. Let Σ be a language on a finite alphabet \mathcal{A} , and let $\Sigma(n)$ be the set of words of length n in Σ . The *complexity* of Σ is the function $f(n) = |\Sigma(n)|$. Let $\varphi(n) = f(n + 1) - f(n)$ and $\psi(n) = \varphi(n + 1) - \varphi(n)$ be the first and second “derivatives” of complexity, respectively.

For any $w \in \Sigma$ let $m_l(w), m_r(w), m_b(w)$ be the number of extensions of w of the type aw, wb, awb respectively. Assume that $m_l(w), m_r(w) \geq 1$ for any $w \in \Sigma$. A word is *bispecial* if $m_l(w), m_r(w) > 1$. Let $\mathcal{B} \subset \Sigma$ be the set of bispecial words, and set $\mathcal{B}(n) = \mathcal{B} \cap \Sigma(n)$. Define the *Cassaigne index* [4] by

$$\mu(w) = m_b(w) - m_l(w) - m_r(w) + 1. \tag{10}$$

Note that $\mu(w) = 0$ if w is not bispecial. Then we have [4]

$$\psi(n) = \sum_{w \in \mathcal{B}(n)} \mu(w) = \sum_{w \in \Sigma(n)} \mu(w). \tag{11}$$

Set $\mu(n) = \sum_{w \in \Sigma(n)} \mu(w)$ for $1 \leq n$ and $\mu(n) = 0$ for $n \leq 0$. Set $M(n) = \sum_{k \leq n} \mu(k)$.

Lemma 3. *The complexity of a language satisfies*

$$f(n) = f(1) + (n - 1)(f(2) - f(1)) + \sum_{k \leq n-2} M(k). \tag{12}$$

Proof. “Integrate” equation (11). \square

Let Σ be the coding language of (X, T, Γ) . For $w \in \Sigma(n)$ let $X(w) \subset X$ be the corresponding 2-cell of Γ_n . Let $\Gamma(w)$ be the restriction of $\Gamma_{-1} \vee \Gamma_{n+1}$ to $X(w)$. Let $V_{\text{ess}}(w)$ (resp. $OE(w)$) be the set of essential vertices (resp. edge overlappings) for $\Gamma(w)$.

Lemma 4. *For any $w \in \Sigma$ we have*

$$\mu(w) = |V_{\text{ess}}(w)| - |OE(w)|. \tag{13}$$

Proof. Let Γ', Γ'' be the restrictions of $\Gamma_{n+1}, \Gamma_{-1}$ to $X(w)$. In equation (2) we have $m_b(w) = |F|, m_r(w) = |F'|, m_l(w) = |F''|, \chi = 1$. Lemma 2 yields the claim. \square

For $n \geq 1$ set

$$V_{\text{ess}}(n) = \bigcup_{w \in \Sigma(n)} V_{\text{ess}}(w), \quad v(n) = |V_{\text{ess}}(n)|, \quad V(n) = \sum_{k \leq n} v(k); \tag{14}$$

$$OE(n) = \bigcup_{w \in \Sigma(n)} OE(w), \quad c(n) = |OE(n)|, \quad C(n) = \sum_{k \leq n} c(k). \tag{15}$$

For $k \leq 0$ set $V(k) = C(k) = 0$. Thus, $v(n)$ (resp. $c(n)$) is the number of essential vertices (resp. edge overlappings) of the join $\Gamma_{-1} \vee \Gamma_{n+1}$.

Proposition 2. *Let (X, T, Γ) be a piecewise convex transformation and set $\mathcal{P}_k = \mathcal{P}(\Gamma_k)$. Then the complexity of (X, T, Γ) satisfies*

$$f(n) = |\mathcal{P}_1| + (n - 1)(|\mathcal{P}_2| - |\mathcal{P}_1|) + \sum_{k \leq n-2} V(k) - \sum_{k \leq n-2} C(k). \tag{16}$$

Proof. Combine Lemma 3 with Lemma 4. □

2. BILLIARD MAP AS A PIECEWISE CONVEX TRANSFORMATION:
THE DICTIONARY

Let M be a complete riemannian surface, and let $P \subset M$ be a connected, compact domain with a piecewise smooth boundary. The billiard flow in P is a particular case of the geodesic flow of a riemannian manifold (with a boundary and corners, in general). The boundary, ∂P , provides the standard cross-section; the corresponding Poincare map is the *billiard in P* . We refer to [16] for details. We will restrict our attention to the case where the curvature $\varkappa = \varkappa(M)$ is constant,³ and ∂P is a finite union of geodesic segments. This is the *billiard in a geodesic polygon* on a surface of constant curvature.

Let \tilde{M} be the universal covering,⁴ let $q: \tilde{M} \rightarrow M$ be the projection, let $F \subset \tilde{M}$ be a fundamental domain, and set $P_F \subset F$ be the preimage of P . The geodesic polygon is *tame* if $q: P_F \rightarrow P$ is a bijective isometry. We will assume that P is tame.⁵ By this assumption, in the discussion that follows $P \subset M$ is a geodesic polygon, where $M = \mathbb{R}^2, \mathbb{H}^2, \mathbb{S}^2$ respectively in our three cases.

We will continue the discussion in a uniform fashion, using the *projective models* of the three geometries at hand. (In a projective model the geodesics are straight lines.) For \mathbb{H}^2 this is the Klein–Beltrami model [2]; a projective model of the elliptic geometry is obtained via the central projection of \mathbb{S}^2 onto a plane. In order to use this model, whenever we consider the elliptic case, we will make the technical assumption that $P \subset \mathbb{S}^2$ is contained in a hemisphere. We will call these spherical polygons *admissible*.

Thus, in all three cases, we represent our geodesic polygon by a euclidean polygon, $P \subset \mathbb{R}^2$. In the hyperbolic case, we have the extra condition $P \subset D$ where D is the unit disc. Let $\mathcal{L} = \mathcal{L}(P)$ be the set of rays intersecting P . The chord structure on \mathcal{L} is induced by the inclusion $\mathcal{L} \subset \mathcal{R}$.⁶ See Examples 2, 3. Let \mathcal{S} (resp. \mathcal{C}) be the set of sides (resp. corners) of P . Let $\Lambda \subset \mathcal{L}$ be the set of rays intersecting \mathcal{C} .

Proposition 3. *Let $P \subset \mathbb{R}^2$ be an arbitrary p -gon, let $\mathcal{L} = \mathcal{L}(P)$ be the set of rays intersecting P , and let $\Lambda = \Lambda(P) \subset \mathcal{L}$ be the set of rays intersecting \mathcal{C} . Then \mathcal{L} is a closed annulus, and (\mathcal{L}, Λ) is a convex cell complex. It has at most $p(p - 1)$ vertices, at most $2p(p - 1)$ edges, and less than $2p(p - 1)$ faces.*

³We normalize $\varkappa = 0, -1, 1$ and refer to these cases as parabolic, hyperbolic and elliptic respectively.

⁴In the three cases at hand $\tilde{M} = \mathbb{R}^2, \mathbb{H}^2, \mathbb{S}^2$.

⁵We make this assumption for simplicity of exposition; our results remain valid, *mutatis mutandis*, without it.

⁶In the hyperbolic case we have $\mathcal{L} \subset \mathcal{R}_{\mathbb{H}} \subset \mathcal{R}$. The chord structures on \mathcal{L} induced from $\mathcal{R}_{\mathbb{H}}, \mathcal{R}$ coincide.

Proof. Let $\tilde{P} \subset \mathbb{R}^2$ be the convex hull of P . Then $\mathcal{L} = \mathcal{L}(\tilde{P})$, and for any bounded, convex domain $\Omega \subset \mathbb{R}^2$ the space $\mathcal{L}(\Omega)$ is a topological annulus [15].

For $o \in \mathcal{C}$ let $\Lambda_o \subset \Lambda$ consist of rays containing o . It is a topological circle, and a union of chords. Thus, $\Lambda = \bigcup_{o \in \mathcal{C}} \Lambda_o$ is a chord graph. Let $F \subset \mathcal{L}$ be a two-cell, i. e., a connected component of $\mathcal{L} \setminus \Lambda$. It suffices to show that for any $l_0, l_1 \in F$ the chord $[l_0, l_1] \subset F$.

For $o \in \mathcal{C}$ and $l \in \mathcal{L} \setminus \Lambda_o$, we define the *sign of o with respect to l* by $\mu_l(o) = 1$ (resp. $\mu_l(o) = -1$) if o is on the left (resp. right) of l . By continuity, for any $o \in \mathcal{C}$ the sign $\mu_l(o)$ is the same for all $l \in F$.

Suppose that $l_0, l_1 \in F$ are anti-parallel, and let $C \subset \mathbb{R}^2$ be the closed strip between them. The equality $\mu_{l_0}(o) = \mu_{l_1}(o)$ holds iff $o \in C$. The absence of corners of P in $\mathbb{R}^2 \setminus C$ implies that $l_0, l_1 \notin \mathcal{L}$, contrary to the assumption. Thus, l_0, l_1 cannot be anti-parallel. Either they intersect or are parallel.

Suppose that l_0, l_1 intersect, and let C be the cone defined in Example 2. Now we have $\mu_{l_0}(o) = \mu_{l_1}(o)$ iff $o \in \mathbb{R}^2 \setminus C$. Thus, C does not contain corners of P . If l_0, l_1 are parallel, we apply the same argument to the strip between l_0, l_1 .

We have shown that $l_0, l_1 \in F$ implies that the rays either intersect or are parallel, and that the cone (resp. strip) between them is free of corners of P . Thus, $[l_0, l_1] \subset F$.

It remains to estimate the numbers of cells of (\mathcal{L}, Λ) . Let v, e, f be the number of vertices, edges, faces respectively. The vertices correspond to the rays $l \in \Lambda$ passing through a pair of points of \mathcal{C} . Since there are $p(p - 1)$ ordered pairs of corners, we have $v \leq p(p - 1)$. The edges belong to the circles $\Lambda_o, o \in \mathcal{C}$. The edges of Λ_o are separated by the vertices corresponding to the pairs $(o, o'), (o', o)$, where $o' \neq o$. Thus Λ_o consists of at most $2(p - 1)$ edges. Since every edge belongs to a unique Λ_o , we have $e \leq 2p(p - 1)$. By Euler's formula, $f = e - v$; hence $f < e$. \square

We will define the billiard phase space by a sequence of steps. A *segment* is an oriented line segment, $x = [b, e), b \neq e$. Let $l \in \mathcal{L}$ be the ray containing $[b, e)$ and having the same direction. A *billiard segment* is a segment such that $[b, e) \subset P$ and $b, e \in \partial P$. Let $X_0 = X_0(P)$ be the set of billiard segments, and set $b = \beta(x), e = \eta(x), l = \lambda_0(x)$. The map $\beta \times \eta \times \lambda_0: X_0 \rightarrow \partial P \times \partial P \times \mathcal{L}$ is injective; from here on we identify X_0 with its image $(\beta \times \eta \times \lambda_0)X_0 \subset \partial P \times \partial P \times \mathcal{L}$. We endow X_0 with the induced topology.

Let $X_1 \subset \partial P \times \partial P \times \mathcal{L}$ be the closure of $(\beta \times \eta \times \lambda_0)(X_0)$. Elements of the boundary $\partial X_0 = X_1 \setminus X_0$ arise from sequences of billiard segments that converge to degenerate limits. The mapping λ_0 extends to X_1 by continuity; we denote this extension by $\lambda_1: X_1 \rightarrow \mathcal{L}$. In the representation $X_1 \subset \partial P \times \partial P \times \mathcal{L}$ the map λ_1 is the projection on the last coordinate. It is continuous and surjective.

A billiard segment may (properly) contain other billiard segments. For $x', x'' \in X_0$ set $x' \sim x''$ iff x', x'' are contained in the same billiard segment, see figure 3. This is an equivalence relation on X_0 ; it extends, by continuity, to X_1 .

Definition 5. The quotient of X_1 by this equivalence relation, endowed with the quotient topology, is the *billiard phase space* $X = X(P)$.

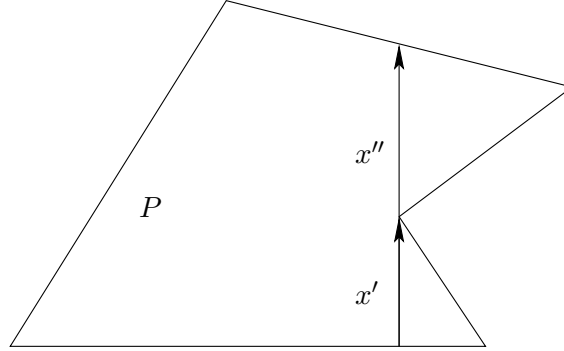


FIGURE 3. Equivalence relation for billiard segments

By definition of the equivalence relation, the mapping $\lambda_1: X_1 \rightarrow \mathcal{L}$ descends to a continuous, surjective mapping $\lambda: X \rightarrow \mathcal{L}$. We will now turn to the billiard map and its inverse.

Let $x_0 = [b, e] \in X_0$ be a billiard segment. Both the billiard map and its inverse are not defined if b, e belong to the same side of P . Also, the billiard (resp. inverse billiard) map $x_0 \mapsto T_{\text{bil}}(x_0) = [e, e_1]$ (resp. $x_0 \mapsto T_{\text{bil}}^{-1}(x_0) = [b_{-1}, b]$) is not defined, if $e \in \mathcal{C}$ (resp. $b \in \mathcal{C}$). Let $\Gamma_0^+ \subset X_0$ (resp. $\Gamma_0^- \subset X_0$) be the former (resp. latter) set. Let $\Gamma_1^+ = \overline{\Gamma_0^+}, \Gamma_1^- = \overline{\Gamma_0^-} \subset X_1$ be the closures of these sets in X_1 . We view $T_{\text{bil}}, T_{\text{bil}}^{-1}$ as self-mappings of X_1 not defined on Γ_1^+, Γ_1^- respectively; besides, they are not defined on $X_1 \setminus X_0$.

Let $q: X_1 \rightarrow X$ be the projection. Set $\Gamma_+ = q(\Gamma_1^+), \Gamma_- = q(\Gamma_1^-)$; let $\Gamma_q = \{x \in X: |q^{-1}(x)| > 1\}$ and $\Gamma_\partial = q(X_1 \setminus X_0)$. Let $T, T^{-1}: X \rightarrow X$ be the push-downs of $T_{\text{bil}}, T_{\text{bil}}^{-1}$ respectively. By definition, T (resp. T^{-1}) is not defined on $\Gamma_+ \cup \Gamma_q \cup \Gamma_\partial$ (resp. $\Gamma_- \cup \Gamma_q \cup \Gamma_\partial$). Set $\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_q \cup \Gamma_\partial$.

Theorem 1. *Let $P \subset M$ be a geodesic polygon on a surface of constant curvature, let $X = X(P)$ be the billiard phase space.*

1. *The pair (X, Γ) is a convex cell complex, and $\lambda: (X, \Gamma) \rightarrow (\mathcal{L}, \Lambda)$ is a (branched) covering of cell complexes.*

2. *The (partially defined) mappings T, T^{-1} yield piecewise convex transformations (X, Γ, T) and (X, Γ, T^{-1}) . The set $\Gamma \subset X$ is the union of the discontinuities of T and T^{-1} .*

Proof. 1. The mapping $\lambda: X \rightarrow \mathcal{L}$ is continuous, surjective, and $\Gamma = \lambda^{-1}(\Lambda)$. Indeed, Λ is the set of rays passing through \mathcal{C} ; by definition, $x \in \Gamma$ iff there is a (possibly degenerate) billiard segment in $q^{-1}(x)$ that contains a corner of P . This holds iff $\lambda(x) \in \Lambda$. Hence $x \in \Gamma$ iff $\lambda(x) \in \Lambda$. By Lemma 3 and Lemma 1, it suffices to show that λ is a branched covering of topological spaces. For $l \in \mathcal{L} \setminus \Lambda$ the intersection $l \cap P$ is a finite disjoint union of billiard segments $x_i = [b_i, e_i], 1 \leq i \leq n(l)$. Projecting them to X , we obtain $n(l)$ phase points; we denote them by x_i as well. Thus, $\lambda^{-1}(l) = \{x_1, \dots, x_{n(l)}\} \subset X \setminus \Gamma$. The number $n(l)$ is constant on connected components of $\mathcal{L} \setminus \Lambda$. Hence, for any two-cell, $G \subset \mathcal{L}$,

we have $\lambda^{-1}(G) = \bigcup_{i=1}^{n(G)} F_i$; the restriction of λ to every F_i is a homeomorphism, $\lambda: F_i \rightarrow G$.

Let now $E \subset \Lambda$ be an edge. A similar argument shows that $\lambda^{-1}(E) = \bigcup_{i=1}^{n(E)} e_i$, a disjoint union, where $e_i \subset \Gamma$, and the restriction of λ to every e_i is a homeomorphism, $\lambda: e_i \rightarrow E$. This verifies the assumptions of Lemma 1, hence the claim.

2. For $x \in X_1$ we denote by $\{x\} \in X$ its equivalence class. Let $G \subset \mathcal{L}$ be a two-cell, and let $F \subset X$ be one of the components of $\lambda^{-1}(G)$. In the proof of claim 1 we have identified F with a subset of X_0 . Let $F = \{x = [b(x), e(x)]\}$. Moreover, all points $b(x)$ (resp. $e(x)$) belong to the interior of a side, $s_{b(F)}$ (resp. $s = s_{e(F)}$) of P ; furthermore, $s_{b(F)} \neq s_{e(F)}$. Let $\sigma_s \in \text{Iso}(M)$ be the geodesic reflection about s . The billiard segments $T_{\text{bil}}(x) = [e(x), e_1)$ are well defined, unless the reflected ray $l_1 = \sigma_s(\lambda(x_0))$ passes through a *nonconvex corner* of P . See Fig. 4. In this case, the billiard segment $T_{\text{bil}}(x_0)$ is not defined, and T_{bil} is discontinuous at x_0 . Nevertheless, the phase point $\{T_{\text{bil}}(x_0)\} = T(x_0) = x_1 \in X$ is well defined. By definition of the quotient topology, T is continuous at x_0 . This proves the continuity of T on F , and hence on $X \setminus \Gamma$.

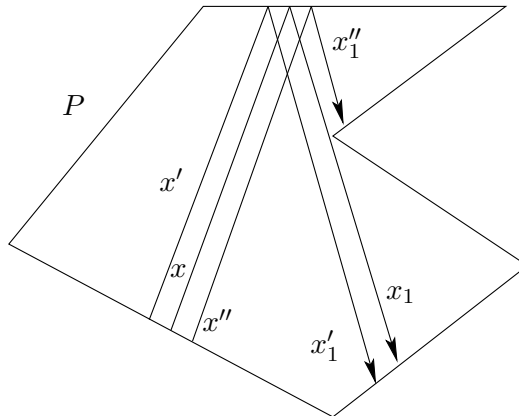


FIGURE 4. Pushing the billiard map down to the quotient phase space; the phase points x'_1, x''_1, x_1 are close to each other

The same argument works for T^{-1} , but we will present an alternative proof. The claim will follow from a special symmetry of the billiard. Let $x_0 = [b, e] \in X_0$. We define the *direction reversing involution* $\rho_0: X_0 \rightarrow X_0$ by $\rho_0(x_0) = [e, b] \in X_0$. It is continuous, and it extends by continuity to $\rho_1: X_1 \rightarrow X_1$, which descends to the involution $\rho: X \rightarrow X$. For $l \in \mathcal{L}$ let l' be the same line with the opposite direction. Then $l \mapsto l'$ defines the involution $r: \mathcal{L} \rightarrow \mathcal{L}$ on the space of rays. The mappings λ_0, λ_1 , and $\lambda: X \rightarrow \mathcal{L}$ commute with the respective involutions. In particular, $\lambda \circ \rho = r \circ \lambda$. The involution ρ_0 satisfies $T_{\text{bil}}^{-1} = \rho_0 \circ T_{\text{bil}} \circ \rho_0$ whenever both sides are defined.

Note that T, T^{-1} are defined and continuous on $X \setminus \Gamma$, and that ρ preserves Γ . Hence, the preceding formula yields

$$\rho \circ T = T^{-1} \circ \rho. \tag{17}$$

We have shown that the discontinuities of both T and T^{-1} belong to Γ . It remains to show the opposite inclusion. Recall that $\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_q \cup \Gamma_\partial$. Neither T nor T^{-1} are defined on Γ_∂ . The map T (resp. T^{-1}) is not defined on Γ_+ (resp. Γ_-). Let $x \in \Gamma_q$, i. e., $|q^{-1}(x)| > 1$. Then there is a billiard segment $[b, e] \in q^{-1}(x)$ such that either $b \in \mathcal{C}$ or $e \in \mathcal{C}$. Hence, either $T_{\text{bil}}([b, e])$ or $T_{\text{bil}}^{-1}([b, e])$ is not defined. Therefore, at least one of T, T^{-1} is not defined on x , yielding a discontinuity.

It remains to check that T, T^{-1} are compatible with the chord structure. Let $F \subset X \setminus \Gamma$ be a two-cell, and let $x_0, x_1 \in F$. Set $G = \lambda(F) \subset \mathcal{L}$, and $l_0 = \lambda(x_0), l_1 = \lambda(x_1) \in G$. Then $\lambda^{-1}(G) = \bigcup_{i=1}^{n(G)} F_i$, and without loss of generality, $F = F_1$. By the proof of claim 1, l_0, l_1 either intersect or are parallel. In either case we have $[l_0, l_1] \subset G$; the rays $l_t \in [l_0, l_1]$ intersect P forming $n(G)$ disjoint curves; each curve is a chord, γ_i , in F_i , $1 \leq i \leq n(G)$. The chord γ_1 joins x_0 with x_1 . Denote by $[b_t, e_t]$ the corresponding family of billiard segments. For $0 \leq t \leq 1$ the points e_t belong to the interior of a side, $s \in \mathcal{S}$. The reflected pencil $\{\sigma_s(l_t) : 0 \leq t \leq 1\}$ is the chord $[\sigma_s(l_0), \sigma_s(l_1)]$. Applying the preceding argument to $[\sigma_s(l_0), \sigma_s(l_1)]$, we see that the intersection of $T(\gamma_1) \subset X$ with any face of X is a chord. Since F is an arbitrary face of X , we have established that the (partially defined on X) mapping T is a piecewise convex transformation (X, Γ, T) . Equation (17) yields the claim for T^{-1} . \square

The following lemma will be used in Section 3.1.

Lemma 5. *Let P be a polygon with p corners, and let (X, Γ) be the corresponding convex cell complex. Then (X, Γ) has at most $2p^3$ faces.*

Proof. A typical ray $l \in \mathcal{L}$ intersects ∂P in at most p points. These points partition l into at most $p + 1$ intervals; at most $p - 1$ of them are billiard segments. Hence, the degree of the covering $\lambda : X \rightarrow \mathcal{L}$ is not greater than $p - 1$. By Proposition 3, the number of faces of \mathcal{L} is less than $2p(p - 1)$. The product $(p - 1) \times 2p(p - 1)$ provides the desired estimate. \square

Let $P \subset M$ be a polygon. Theorem 1 associates with the billiard map of P a piecewise convex transformation (X, Γ, T) . We will establish a dictionary between the billiard in P and (X, Γ, T) . In particular, we will interpret properties of billiard orbits in terms of the notions introduced in Section 1.

A billiard segment $s = [b, e]$ is *regular* (resp. *singular*) if it does not contain (resp. contains) corners of P . A billiard orbit is a finite sequence of billiard segments; their number is the *length* of the orbit. Let s_0, s_1, \dots, s_n be a sequence of billiard segments. It is a billiard orbit iff the segments $s_i, 0 < i < n$, are regular and $s_{i+1} = T_{\text{bil}}(s_i), 0 \leq i < n$. To distinguish billiard orbits from arbitrary sequences of billiard segments, we will use the notation like $\omega = (s_0, s_1, \dots, s_n)$. The orbit (s_0, s_1, \dots, s_n) is regular (resp. singular; resp. strongly singular) if the segments s_0, s_n are also regular (resp. at least one of s_0, s_n is singular, resp. both s_0, s_n are singular).

Analogously, a sequence $(x_0, x_1, \dots, x_n) \subset X$ is a regular (resp. singular, resp. strongly singular) *phase orbit* if all x_i are regular and $T(x_i) = x_{i+1}, 0 \leq i \leq n$ (resp. either x_0 or x_n is singular, and $x_0 = T^{-1}(x_1)$ in the former case; resp. both x_0, x_n

are singular, with the convention above). We will also say that (x_0, x_1, \dots, x_n) is an *orbit of* (X, Γ, T) .

A *generalized diagonal* $\omega = (s_0, s_1, \dots, s_n)$ is a strongly singular billiard orbit such that $b(s_0)$ (resp. $e(s_n)$) is a corner of P , and neither segment contains other corners.

Let $\omega(0) = (s_0(0), s_1(0), \dots, s_n(0))$ and $\omega(1) = (s_0(1), s_1(1), \dots, s_n(1))$ be two billiard orbits of the same length. Suppose that for $0 \leq i \leq n$ the chord $\gamma_i = [s_i(0), s_i(1)]$ exists, and for $0 \leq i \leq n-1$ we have $T_{\text{bil}}(\gamma_i) = \gamma_{i+1}$. This defines the *chord of billiard orbits* $[\omega(0), \omega(1)]$ joining $\omega(0), \omega(1)$.

Let $\omega(0) = (s_0(0), s_1(0), \dots, s_n(0))$ and $\omega(1) = (s_0(1), s_1(1), \dots, s_n(1))$ be generalized diagonals such that the chord $[\omega(0), \omega(1)]$ exists, and $[\omega(0), \omega(1)] = \{\omega(t) : 0 \leq t \leq 1\}$, where all $\omega(t)$ are generalized diagonals. Then $[\omega(0), \omega(1)]$ is a *chord of generalized diagonals*.

Definition 6. A generalized diagonal ω is *isolated* if it is not contained in a chord of generalized diagonals.

A chord is *maximal* if it is not properly contained in a longer chord. Every chord of generalized diagonals is contained in a unique *open maximal chord* $\{\omega(t) : 0 < t < 1\}$. Since the limits $\omega(0), \omega(1)$ contain corners in their interior, they are not generalized diagonals.

In what follows we refer to the correspondence established in the discussion above as the *dictionary* (between billiard orbits and phase space orbits).

Proposition 4. Let $P \subset M$ be a geodesic polygon on a surface of constant curvature, let X be the billiard phase space, and let (X, Γ, T) be the associated piecewise convex transformation. Let $n \geq 1$. Then the following holds:

1. The dictionary establishes a bijection between the set $V_{\text{ess}}(n) \subset X$ and the set of isolated generalized diagonals of length $(n+2)$ in P ;
2. The dictionary establishes a bijection between the set $OE(n) \subset X$ and the set of maximal chords of generalized diagonals of length $(n+2)$ in P .

Proof. Let $x_0 \in X \setminus \Gamma$. Then x_0 does not contribute to $V_{\text{ess}}(n)$ or to $OE(n)$ unless the following conditions hold:

- (i) The points $x_i = T^i(x_0) \in X \setminus \Gamma$ for $1 \leq i \leq n-1$;
- (ii) The points $x_n = T^n(x_0), x_{-1} = T^{-1}(x_0) \in \Gamma$.

Suppose that these conditions hold. Denote by $\text{ch}_+ \subset \Gamma$ (resp. $\text{ch}_- \subset \Gamma$) the chord containing x_n (resp. x_{-1}). Let $\gamma_n \subset T^{-n}(\text{ch}_+)$ (resp. $\gamma_{-1} \subset T(\text{ch}_-)$) be the maximal chords contained in the respective pull-backs, and containing x_0 . Then $x_0 \in V_{\text{ess}}(n)$ (resp. x_0 contributes to $OE(n)$, increasing $OE(n)$ by 1) iff the chords γ_{-1}, γ_n intersect transversally (resp. the chords γ_{-1}, γ_n overlap).

By Theorem 1, x_0, \dots, x_{n-1} defines a regular billiard orbit, (s_0, \dots, s_{n-1}) . The billiard segments $s_{-1} = T_{\text{bil}}^{-1}(s_0), s_n = T_{\text{bil}}(s_{n-1})$ are singular. By choosing them appropriately, we can assume without loss of generality that s_{-1} (resp. s_n) begins (resp. ends) in a corner.

Let $p_{-1}(t), -\alpha < t < \beta$, and $q_n(t), -\gamma < t < \delta$, be the chords of billiard segments corresponding to ch_-, ch_+ respectively. We normalize the parameter t so that $p_{-1}(0) = s_{-1}, q_n(0) = s_n$. Applying the billiard map, we obtain two

chords of billiard orbits: $(p_{-1}(t), p_0(t))$, $-\alpha < t < \beta$, and $(q_0(t), q_1(t), \dots, q_n(t))$, $-\gamma < t < \delta$. By construction, $p_0(0) = q_0(0) = s_0$. We denote the two chords of billiard orbits by $[p]$, $[q]$ respectively.

The phase space cord $\gamma_{-1} \subset T(\text{ch}_-)$ (resp. $\gamma_n \subset T^{-n}(\text{ch}_+)$) corresponds to the billiard segment chord $p_0(t)$, $-\alpha < t < \beta$, (resp. $q_0(t)$, $-\gamma < t < \delta$). The chords γ_{-1}, γ_n are transversal at x_0 iff the billiard segment chords $p_0(t)$, $-\alpha < t < \beta$, $q_0(t)$, $-\gamma < t < \delta$, that coincide at $t = 0$, have no other common billiard segments. Equivalently, the billiard orbit chords $[p]$, $[q]$ fit together forming the generalised diagonal ω only at $t = 0$, and not for $t \neq 0$. This happens iff ω is isolated.

The phase space chords γ_{-1}, γ_n overlap at x_0 iff we have $p_0(t) = q_0(t)$ for $-\varepsilon < t < \varepsilon$. Equivalently, the billiard orbit chords $[p]$, $[q]$ fit together for $-\varepsilon < t < \varepsilon$, forming a chord of generalised diagonals. \square

In view of Proposition 4, whenever we talk of strongly singular billiard orbits in the context of billiard complexity, we restrict our attention to orbits of length at least three.

3. COMPLEXITY OF THE BILLIARD IN A GEODESIC POLYGON

We will now apply the preceding material to the complexity of billiards in geodesic polygons on surfaces of constant curvature. First, we consider the three cases simultaneously, emphasizing their similarities.

3.1. Arbitrary curvature. Let $P \subset M$ be a geodesic polygon. Let $G = G(P) \subset \text{Iso}(M)$ be the group generated by the geodesic reflections in the sides of P . Any $g \in G$ is represented by a word whose letters are the generators; the *length* is the minimal number of letters [7]. Let $G^{(n)} \subset G$ be the set of elements of length at most n . Then $G^{(n)} \subset G^{(n+1)}$, and $G = \bigcup_{n=0}^{\infty} G^{(n)}$.

The operation of *unfolding* sends billiard orbits in P into geodesics in M [16]. See Fig. 5. Let $\gamma = (x_0, x_1, \dots, x_n)$ be a billiard orbit. Its unfolding is the geodesic $\tilde{\gamma} = (x_0, \tilde{x}_1, \dots, \tilde{x}_n)$; there is $g = g(\gamma) \in G^{(n)}$ such that $\tilde{x}_n = g \cdot x_n$.

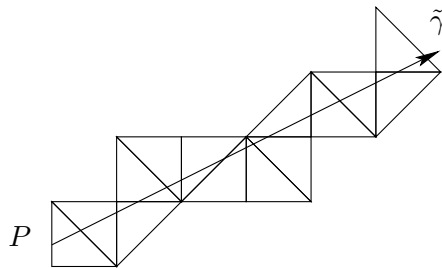


FIGURE 5. Unfolding a billiard trajectory

Lemma 6. 1. If $\varkappa(M) \leq 0$, then all generalized diagonals in P are isolated.
 2. Let $\varkappa(M) > 0$. Let $\gamma = (x_0, x_1, \dots, x_n)$ be a generalized diagonal, and let $c_0, c_n \in \mathcal{C}$ be its endpoints. Let $\tilde{\gamma} = (x_0, \tilde{x}_1, \dots, \tilde{x}_n)$ be the unfolding of γ , and let

$\tilde{c}_n = g(\gamma) \cdot c_n \in M$ be the endpoint of $\tilde{\gamma}$. Then γ belongs to a chord of generalized diagonals iff the points c_0, \tilde{c}_n coincide or are antipodal.

Proof. Let $\gamma(t) = (x_0(t), x_1(t), \dots, x_n(t))$, $-\varepsilon \leq t \leq \varepsilon$, be a chord of generalized diagonals, and let $\tilde{\gamma}(t) = (x_0(t), \tilde{x}_1(t), \dots, \tilde{x}_n(t))$, $-\varepsilon \leq t \leq \varepsilon$, be the corresponding beam of geodesics. The elements of G produced by the unfolding do not depend on the chord parameter. In particular, $g(\gamma(t)) = g$ for all t . Thus, all geodesics $\tilde{\gamma}(t)$ begin at c_0 and end at $g \cdot c_n$.

The preceding argument is reversible. Therefore, a family $\gamma(t)$ of $(n+1)$ -segment generalized diagonals is a chord iff the unfolded family $\tilde{\gamma}(t)$ is a beam of geodesics in M , emanating from $c_0 \in \mathcal{C}$ and refocusing at $g \cdot c_n \in M$. If $\varkappa \leq 0$, this is impossible, implying claim 1. Let $\varkappa > 0$, i. e., M is the sphere. A beam emanating from a point refocuses at the antipodal point and also at the point itself. \square

We denote by $\text{gd}(k)$ (resp. $\text{cgd}(k)$) the number of isolated (resp. maximal chords of) k -segment generalized diagonals in P . Set

$$\text{GD}(n) = \sum_{3 \leq k \leq n} \text{gd}(k), \quad \text{CGD}(n) = \sum_{3 \leq k \leq n} \text{cgd}(k).$$

Thus, $\text{GD}(n)$ (resp. $\text{CGD}(n)$) is the number of isolated (resp. maximal chords of) generalized diagonals of length at most n .

Proposition 5. *Let $P \subset M$ be a geodesic polygon, and let \varkappa be the curvature of M . Let (X, T, Γ) be the piecewise convex transformation associated with the billiard in P , and let $f_\Gamma(\cdot)$ be the corresponding complexity. Then there are integers q_1, q_2 depending on P , so that the following holds.*

1. For $\varkappa = 1$ we have

$$f_\Gamma(n) = q_1 + q_2 n + \sum_{k \leq n} \text{GD}(k) - \sum_{k \leq n} \text{CGD}(k). \tag{18}$$

2. If $\varkappa \leq 0$, then

$$f_\Gamma(n) = q_1 + q_2 n + \sum_{k \leq n} \text{GD}(k). \tag{19}$$

Proof. By Theorem 1 and Proposition 4, equation (16) yields the first claim. Combining it with Lemma 6, we obtain equation (19). \square

It is convenient to code billiard orbits in P by the sides that they visit. This is the *side coding*, and the corresponding complexity is the *side complexity*, $f_{\text{side}}(n)$. We will define it precisely. Recall that \mathcal{S} is the set of sides of P ; we use the notation $a, b, c, \dots \in \mathcal{S}$. If $\omega = (s_1, \dots, s_n)$ is a regular n -segment billiard orbit, let $a_0 = b(s_1), a_1 = b(s_2), \dots, a_n = e(s_n) \in \mathcal{S}$ be the sides that ω visited. The word $a_0 a_1 \dots a_n \in \Sigma_{\text{side}}(n+1)$ is a side code. We set $f_{\text{side}}(n) = |\Sigma_{\text{side}}(n+1)|$.

Proposition 6. *Let $P \subset M$ be a geodesic polygon, and let \varkappa be the curvature of M .*

1. If $\varkappa = 1$, then there are integers q_1, q_2 (depending on P) such that

$$f_{\text{side}}(n) \leq q_1 + q_2 n + \sum_{k \leq n} \text{GD}(k) - \sum_{k \leq n} \text{CGD}(k). \tag{20}$$

2. If $\varkappa \leq 0$, then there are constants c_1, c_2 such that

$$f_{\text{side}}(n) \leq c_1 + c_2 n + \sum_{k \leq n} \text{GD}(k). \tag{21}$$

3. If P is a convex polygon, then equations (20, 21) are equalities.

Proof. Denote by $X_0(a, b) \subset X_0$ the set of regular billiard segments $[p, q]$ such that $p \in a, q \in b$. Let $X(a, b) \subset X$ be the corresponding subset of the phase space. Every face of Γ is contained in some $X(a, b)$. Thus, any code by the faces of Γ defines a side code, yielding a surjection $\Sigma(n) \rightarrow \Sigma_{\text{side}}(n + 1)$. Therefore $f_{\text{side}}(n) \leq f_{\Gamma}(n)$. Let now P be convex. The discussion of Section 2 shows that $\lambda: X \rightarrow \mathcal{L}$ is an isomorphism of convex cell complexes; the faces of Γ are the sets $X(a, b) \subset X$, where $a, b \in \mathcal{S}$ are different sides of P . Thus, $f_{\text{side}}(n) = f_{\Gamma}(n)$. Now Proposition 5 implies all three claims. \square

We will now establish a connection between the complexity of billiard as a piecewise convex transformation and the side complexity.

Proposition 7. *Let $P \subset M$ be a geodesic polygon with p corners on a surface of constant curvature. Let $f_{\Gamma}(\cdot)$ be the complexity of the piecewise convex transformation (X, Γ, T) associated with the billiard map of P ; let $f_{\text{side}}(n)$ be the side complexity of the billiard map. Then $f_{\text{side}}(n) \leq f_{\Gamma}(n) \leq 2p^3 n^3 f_{\text{side}}(n)$.*

Proof. We have established the left inequality in the proof of Proposition 6. Let \mathcal{Q} (resp. \mathcal{P}) be the side partition of X (resp. the defining partition of (X, T, Γ)). Let \mathcal{Q}_n and $\mathcal{P}_n, 1 \leq n$, be the respective sequences of partitions. Then $\mathcal{P}_n \prec \mathcal{Q}_n$, and $f_{\text{side}}(n) = |\mathcal{Q}_n|, f_{\Gamma}(n) = |\mathcal{P}_n|$.

If A is an atom of \mathcal{Q}_n , let $r_n(A)$ be the number of atoms of \mathcal{P}_n that partition A , and set $r_n = \max_{A \in \mathcal{Q}_n} r_n(A)$. Then $f_{\Gamma}(n) \leq r_n f_{\text{side}}(n)$.

A point in A defines an n -segment billiard orbit in P , and the mapping is injective. We will therefore speak of n -segment billiard orbits as points in atoms of \mathcal{Q}_n . Let $\gamma \in A$ be an n -segment billiard orbit. The ‘‘polygon’’ \tilde{P}_A obtained by unfolding P along γ does not depend on the choice of γ . Although \tilde{P}_A is not a polygon, in general, the basic concepts of polygonal billiard apply to it.⁷

Let $(\tilde{X}_A, \tilde{T}_A, \tilde{\Gamma}_A)$ be the piecewise convex transformation associated with the billiard in \tilde{P}_A . The unfolded orbits $\tilde{\gamma}', \tilde{\gamma}''$ belong to the same cell of $\tilde{\Gamma}_A$ iff γ', γ'' belong to the same atom in the partition of A induced by \mathcal{P}_n .

Thus, $r_n(A)$ is less than or equal to the number of faces in $\tilde{\Gamma}_A$. By construction, \tilde{P}_A has at most pn corners. Hence, by Lemma 5, $r_n(A) \leq 2p^3 n^3$. Since A is arbitrary, $r_n \leq 2p^3 n^3$. \square

3.2. The euclidean case. Billiard dynamics in euclidean polygons is a classical subject. Still, many basic questions remain open [10], [16]. The side complexity is subexponential [11];⁸ it is plausible that it is in fact polynomial. Although this is open for general polygons, it has been proved for *rational polygons*. A euclidean

⁷ If P is convex, then $\tilde{P}_A \subset \mathbb{R}^2$ is a polygon. In general, \tilde{P}_A is a flat surface with a boundary; the reader may think of \tilde{P}_A as a *polygon in a (branched) covering of \mathbb{R}^2* .

⁸Implying that the billiard in any euclidean polygon has zero topological entropy.

polygon is *rational* if its angles are rational multiples of π . The billiard in rational polygons is of interest on its own; it has also been instrumental in the study of billiard dynamics in generic polygons [14].

Theorem 2. *Let $P \subset \mathbb{R}^2$ be a rational euclidean polygon.*

1. *There exists $c = c(P) > 0$ such that*

$$f_{\text{side}}(n) < cn^3. \quad (22)$$

2. *Suppose, in addition, that P is convex. Then there exist positive numbers c_1, c_2 such that*

$$c_1 n^3 < f_{\text{side}}(n) < c_2 n^3. \quad (23)$$

Proof. By a theorem of Masur [14], for any rational polygon there exist positive constants c'_1, c'_2 such that

$$c'_1 n^2 < \text{GD}(n) < c'_2 n^2.$$

Proposition 6 implies inequality (22) and both inequalities (23). \square

3.3. The elliptic case. The theorem below holds for all spherical polygons P ; for simplicity of exposition, we will prove it under the assumption that P is an admissible polygon.

Theorem 3. *For any spherical polygon the side complexity is subexponential.*

Proof. Let $\text{Geo}(\mathbb{S}^2)$ be the space of oriented great circles, and let $\varphi: \text{Geo}(\mathbb{S}^2) \rightarrow \mathbb{S}^2$ be the standard diffeomorphism.⁹ It endows $\text{Geo}(\mathbb{S}^2)$ with an invariant riemannian metric. The geodesics in this metric are the chords in $\text{Geo}(\mathbb{S}^2)$. Let (X, T, Γ) be the associated piecewise convex transformation. By the proof of Lemma 3, the branched covering $\lambda: X \rightarrow \text{Geo}(\mathbb{S}^2)$ induces metrics on the faces of Γ ; every face is a convex polygon.

The billiard map T_{bil} is a piecewise geodesic reflection; hence (X, T, Γ) is a piecewise isometry on a convex partition. By Theorem 4.2 of [11], $f_{\Gamma}(n)$ has subexponential growth. Proposition 7 implies the claim. \square

By Lemma 6, a spherical polygon may have chords of generalised diagonals. The examples below illustrate other special features of the spherical billiard.

Example 4. Let $P \subset \mathbb{S}^2$ be a polygon such that $G(P)$ is a finite group.¹⁰ Then every billiard orbit in P is periodic. Since the prime periodic orbits yield a finite number of symbolic codes, f_{side} is bounded.

Example 5. Let P_{α} be the “bigon”, bounded by two geodesics, a, b connecting the North and the South poles, where α is the angle between them. (Note that bigons are not admissible polygons.) For any n the set $\Sigma_{\text{side}}(n)$ consists of 2 elements: $abab\dots$ and $baba\dots$. Thus, $f_{\text{side}}(n) = 2$. For any α the equator provides a 2-segment periodic orbit in P_{α} . If α is π -rational, then P_{α} fits Example 4, hence every regular orbit is periodic.

⁹See Section 4.2 for details.

¹⁰These polygons are classified. See, e. g., [6].

Claim. *Let α be π -irrational. Then the equator orbit is the only prime periodic orbit in P_α .*

Proof. Denote by ρ_α the rotation by α about the z -axis. Let γ be a periodic orbit. We can assume that it has an even number, $2m$, of segments, and that its symbolic code is $ba\dots ba$. Then the isometry $g(\gamma)$ obtained by unfolding γ is $\rho_\alpha^{2m} \neq 1$.

Let $\ell(\gamma)$ be the spherical geodesic corresponding to γ . (Note that $\ell(\gamma)$ differs, in general, from the unfolding $\tilde{\gamma}$ which is a geodesic segment along $\ell(\gamma)$.) By periodicity of γ , $\ell(\gamma)$ is invariant under $g(\gamma)$. The only geodesic invariant under ρ_α^{2m} is the equator. \square

By convention, a periodic billiard orbit in P does not pass through its corners. In particular, it cannot trace the boundary of P . It is not known whether every euclidean polygon has a periodic orbit [10]. Our next example is a spherical polygon without periodic orbits.

Example 6. For $0 < \alpha < 2\pi$ let $Q = Q_\alpha$ be the isosceles spherical triangle with angle α and two right base angles.

Claim. *If α is π -rational then every billiard orbit in Q is periodic. If α is irrational, then Q has no periodic billiard orbits.*

Proof. The bigon P of Example 5 is obtained by doubling Q about the equator. Every billiard orbit, γ , in Q lifts to a billiard orbit $\tilde{\gamma}$ in P ; the orbit γ is periodic iff so is $\tilde{\gamma}$. If α is π -rational, the claim holds, by Example 4. Let α be irrational, and let γ be a periodic orbit in Q . Since, by Example 5, $\tilde{\gamma}$ runs along the equator, γ traces the boundary of Q . \square

3.4. The hyperbolic case. It is not surprising that the complexity of the billiard in a hyperbolic polygon grows exponentially. We will obtain a more precise estimate. A positive function, $s(\cdot)$, of natural argument is *temperate* if for any $h > 0$ and all sufficiently large n we have $e^{-hn} < s(n) < e^{hn}$.

Theorem 4. *Let $P \subset \mathbb{H}^2$ be a geodesic polygon, and let h_{top} be the topological entropy of the billiard in P . Let $f_{\text{side}}(\cdot)$ be the side complexity of the billiard map in P . Then $h_{\text{top}} > 0$; there exists a temperate function $s(\cdot)$ such that $f_{\text{side}}(n) = s(n)e^{h_{\text{top}}n}$.*

Proof. The billiard flow of P is (uniformly) hyperbolic [9].¹¹ Thus, the metric entropy of the billiard flow (with respect to the Liouville measure) is positive. By Abramov's formula [1], the metric entropy of the billiard map in P (with respect to the canonical measure) is positive as well. By the maximum principle, $h_{\text{top}} > 0$.

Let (X, T, Γ) be the piecewise convex transformation associated with the billiard in P , and let $\mathcal{P} = \mathcal{P}(\Gamma)$ be the corresponding convex partition of X . Let \mathcal{Q} be the (nonconvex, in general) partition of X defined by the sides of P .

Let $\alpha(t), \beta(t)$ be infinite billiard orbits that visit the same sides of P for $-\infty < t < \infty$. Then their unfoldings, $\tilde{\alpha}(t), \tilde{\beta}(t)$ are infinite geodesics in \mathbb{H}^2 ; the distance between $\tilde{\alpha}(t), \tilde{\beta}(t)$ is bounded, as $-\infty < t < \infty$. Hence $\tilde{\alpha}(-\infty) = \tilde{\beta}(-\infty)$ and

¹¹This is a special case of a more general result in [9]. It can also be obtained directly.

$\tilde{\alpha}(\infty) = \tilde{\beta}(\infty)$ implying $\tilde{\alpha} = \tilde{\beta}$. Therefore, $\alpha = \beta$. Hence \mathcal{Q} is a generating partition.

Since $\mathcal{P} \prec \mathcal{Q}$, the partition \mathcal{P} is generating as well. By [11], the complexity of (X, T, Γ) satisfies $f_\Gamma(n) = s_1(n)e^{h_{\text{top}}n}$, where s_1 is a temperate function. The claim follows, by Proposition 7. \square

4. COMPLEXITY OF THE OUTER BILLIARD

Piecewise convex transformations (X, Γ, T) , where X is a metric space, the chords are geodesics, and the restrictions of T to 2-cells are isometries are the *piecewise convex isometries*. This is a wide class of piecewise isometries; there are many other classes, e.g., interval exchanges [14].

We will investigate a particular example of piecewise convex isometries – the outer billiard transformation. Let $P \subset M$ be a convex geodesic p -gon, where M is a simply connected surface of constant curvature \varkappa . For $x \in M$ denote by $T_x: M \rightarrow M$ the geodesic symmetry about x . Let a, b, c, \dots be the corners of P listed counterclockwise. If $\varkappa = 0, -1$, set $X = X(P) = M \setminus P$. If $\varkappa = 1$ (i.e., $M = \mathbb{S}^2$), let P' be the antipodal polygon,¹² and set $X = X(P) = M \setminus \{P \cup P'\}$.

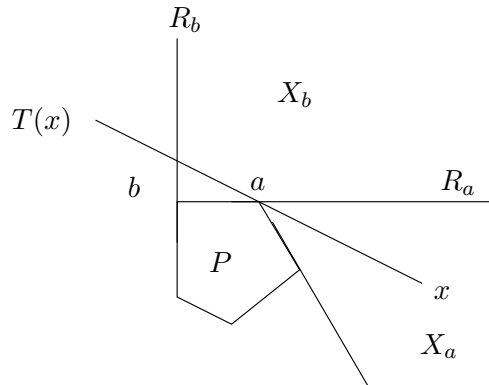


FIGURE 6. Definition of the outer billiard map

For a corner, say a , of P , let $R_a \subset X$ be the geodesic ray extending the side ab in the direction of a . The set $X \subset M$ is a metric space, the chords are the geodesics in X , and the chord graph $\Gamma = R_a \cup R_b \cup \dots$ is convex. Set $\mathcal{P} = \mathcal{P}(\Gamma)$, and let X_a, X_b, \dots be the closed 2-cells. See figure 6.

Definition 7. The *outer billiard* about P is the piecewise convex isometry (X, Γ, T) such that the restrictions $T|_{X_a}, T|_{X_b}, \dots$ are the geodesic symmetries $T_a: X_a \rightarrow X, T_b: X_b \rightarrow X, \dots$. The space $X \subset M$ is the *phase space of the outer billiard*.

Remark 2. If $P \subset \mathbb{S}^2$, then $P \cup P'$ is the set of points that are not contained in great circles supporting P . This implies that $X = \mathbb{S}^2 \setminus \{P \cup P'\}$ is invariant under the outer billiard transformation.

¹²In this section we assume, as before, that P is an admissible polygon.

We will use the notation $T: X \rightarrow X$ for the outer billiard.¹³ The *complexity of the outer billiard* is the complexity of (X, Γ, T) with respect to the partition \mathcal{P} .

If G is a group with a finite set $S = \{s_1, \dots, s_p\}$ of generators, we denote by $G_S^{(n)} \subset G$ the set of elements that can be represented by products of at most n elements of S and their inverses. The growth class of the sequence $g_S(n) = |G_S^{(n)}|$ does not depend on the choice of S [7]. If $g_S(n) \sim n^d$, then we say that the group G grows polynomially, with degree d . In our case, $G = G(P) \subset \text{Iso}(M)$ is the group generated by the set $S = \{T_a, T_b, \dots\}$ of geodesic symmetries about the corners of P . We proceed to study the three cases at hand.

4.1. The euclidean case. We will obtain polynomial bounds on the complexity of outer billiard.

Theorem 5. *Let P be a convex euclidean p -gon, and let $f(\cdot)$ be the complexity of the outer billiard about P . Then $n = O(f(n))$, $f(n) = O(n^{p+1})$.*

Proof. The edges of the graph Γ_n are parallel to the sides of P ; each edge is a segment or a half-line. Assume, for simplicity of exposition, that P has no parallel sides. Then there are p directions. For each direction there are n parallel half-lines, hence their total number is pn . Since they partition X into pn components, the number of faces of Γ_n is at least pn . This proves a linear lower bound on complexity.¹⁴ To prove the upper bound, we will need a few lemmas.

Let $G = G(P)$, and let $S = \{T_1, \dots, T_p\}$ be the natural set of generators.

Lemma 7. *The group $G = G(P)$ grows at most polynomially, with degree $p - 1$.*

Proof. The subgroup $H \subset G$ generated by $T_1T_p, T_2T_p, \dots, T_{p-1}T_p$ is a quotient of \mathbb{Z}^{p-1} ; hence its growth is bounded by n^{p-1} . Since H is a normal subgroup of G of index 2, the two groups have the same growth. \square

Let $\Gamma = \partial\mathcal{P}$, and let $\Gamma_1, \Gamma_2, \dots$ be the canonical sequence of graphs; see Section 1. Let γ_n be the set of edges of $\Gamma_n \setminus \Gamma_{n-1}$.

Lemma 8. *The first difference of the sequence $|\gamma_n|$ is bounded by n^{p-1} .*

Proof. The edges of γ_{n+1} are obtained from the edges of γ_n by applying T^{-1} . Each time a singularity half-line of T^{-1} intersects an edge of γ_n , this edge splits into two, and thus contributes 1 to $|\gamma_{n+1}| - |\gamma_n|$.

Let L_n be the set of straight lines obtained by reflecting at most n times in the corners the extensions of the sides of P . By Lemma 7, $|L_n| = O(n^{p-1})$. Each of these lines intersects a singularity half-line of T^{-1} at most once, therefore the total number of intersections of the lines in L_n with the singularity half-lines of T^{-1} is bounded above by n^{p-1} . The edges of γ_n belong to the lines from L_n , therefore the total number of intersections of these edges with the singularity half-lines of T^{-1} is bounded above by n^{p-1} . (Note that the number of edges of γ_n could be bigger.) \square

¹³If a danger of confusion with the inner billiard arises, we will use the superscript, $T^{\text{out}}: X \rightarrow X$.

¹⁴We conjecture that there is a quadratic lower bound.

We will now estimate the number of faces of Γ_n . Denote by $|F_n|, |E_n|, |V_n|$ the number of faces, edges, vertices of the graph Γ_n respectively. By Lemma 8, the growth of the second difference of the sequence $|E_n|$ is at most polynomial of degree $p - 1$, hence $|E_n| = O(n^{p+1})$. The edges of Γ_n are parallel to the sides of P , thus may have at most p possible directions. Therefore, each face of Γ_n is at most a $2p$ -gon, and the valence of each vertex of Γ_n is at most $2p$. Thus, $|E_n| \leq p|F_n|$, $|E_n| \leq p|V_n|$. Euler's formula $|V_n| - |E_n| + |F_n| = 0$ implies $p|F_n| \leq (p - 1)|E_n|$, hence $|F_n| = O(|E_n|)$. This completes the proof of Theorem 5. \square

We have assumed in the proof of Theorem 5 that the abelian group generated by the sides of P has the maximal rank, $p - 1$. Although this holds generically, the assumption is not necessary. Our argument proved the statement below.

Corollary 2. *Let P be a convex euclidean p -gon, and let $r \leq p - 1$ be the rank of the abelian group generated by translations in the sides of P . Then the complexity of the outer billiard about P is $O(n^{r+2})$.*

A polygon is *rational* if the rank above is 2. Rational polygons are dense in the space of all polygons. We will study complexity of the outer billiard about a rational polygon.

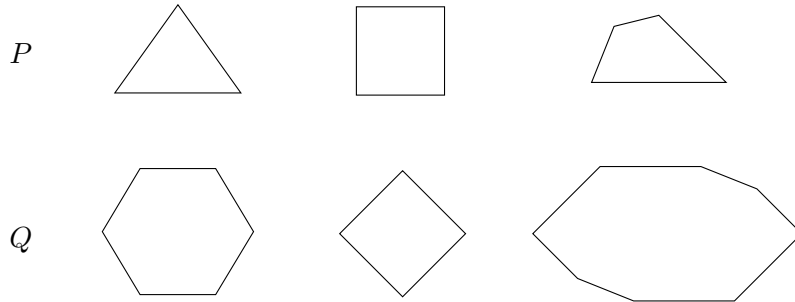


FIGURE 7. Outer billiard; examples of polygons P and Q

We regard the plane as a vector space, with the center in the interior of the convex p -gon P . A well known construction [16] associates with P a homothetic family of centrally symmetric convex polygons with at most (resp. exactly) $2p$ sides (resp. if P is a generic p -gon). Let Q be a particular polygon in this family. Each of its sides is parallel to a diagonal of P . See Fig. 7. We endow the plane with the Minkowski norm $|\cdot|$ such that Q is the unit disc. The radius, etc will be understood with respect to $|\cdot|$. We set $Q(r) = r \cdot Q$.

The polygon Q determines the geometry of orbits of T^2 “at infinity” [16]. We will elaborate. Let x be a point in the plane which is sufficiently far from the origin. Let Q_x be the circle centered at the origin and passing through x . Let $a \subset Q_x$ be the side containing x , and let d be the corresponding diagonal of P (parallel to a). Then T^2 translates x along a by $2|d|$; this continues until the orbit of x overshoots a . Let $y = T^{2m}$ be the corresponding point. Then the same recipe is applied to y , etc. See Fig. 8.

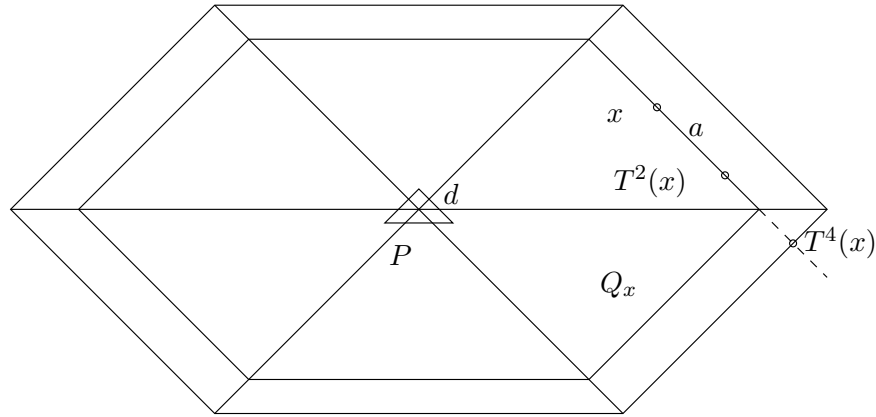


FIGURE 8. Second iteration of the outer billiard map “at infinity”

Let a be an arbitrary side of Q , and let d be the corresponding diagonal of P . The polygon P is *quasirational* if, up to a common factor, the p numbers $r_a = |a|/|d|$ are rational.

Theorem 6. *Let P be a rational polygon, and let $f(\cdot)$ be the complexity of the outer billiard about P . Then $f(n) \sim n^2$.*

Proof. Every rational polygon is quasirational. By a construction of R. Kolodziej [13],¹⁵ there is a nested sequence of T -invariant, polygonal, simply connected domains $\dots \subset U_i \subset U_{i+1} \subset \dots$ exhausting the plane. By [13], there exists a constant $C = C(P) > 0$ such that the *Kolodziej domains* satisfy $Q(Ci) \subset U_i \subset Q(C(i + 1))$.

Lemma 9. *Let P be an arbitrary convex polygon, and let $f(\cdot)$ be the complexity of the outer billiard about P . There exists $C_1 > 0$ such that the contribution to $f(n)$ of the exterior of the disc of radius C_1n grows linearly.*

Proof. We will use the preceding notation and terminology. Let $C_1 > 2/r_a$ for all sides of P .

Consider the T^2 -orbit of length n of an arbitrary point x outside of $Q(C_1n)$. It follows a side, a , of Q for $k \leq n$ iterations, then it “jumps” to the adjacent side, a' , and follows it for $n - k$ iterations. Counting the possibilities (and assuming that Q is a $2p$ -gon, which is the generic situation) we obtain $2p(n + 1)$ types of T^2 -orbits of length n . But different types mean different contributions to $f(2n)$, and vice versa. \square

Since P is a rational polygon, the group $G \subset \text{Iso}(\mathbb{R}^2)$ is discrete. The graphs Γ_n are obtained from a finite collection of half-lines by G -action, hence $\Gamma_\infty = \bigcup_{n \geq 1} \Gamma_n$ belongs to a discrete collection of lines. Therefore Γ_∞ is a graph, and the sequence $\Gamma_1 \subset \dots \subset \Gamma_n \subset \dots$ stabilizes on compacta. Moreover, there is a finite collection of convex polygons, such that every face of Γ_∞ is congruent to a polygon in this

¹⁵Using it, Kolodziej proved that the outer billiard orbits for quasirational polygons are bounded [13]. For general polygons this question is open [16].

collection. Hence the areas of the faces of Γ_∞ are bounded away from zero and infinity.

Note that the constant C_1 in Lemma 9 can be chosen arbitrarily large. We choose it so that $\frac{C_1}{C} = \tau \in \mathbb{N}$. Then for all n sufficiently large

$$Q(C_1n) \subset U_{\tau n} \subset Q(C_1n + C). \tag{24}$$

By Lemma 9, up to a linear term, $f(n)$ is the number of faces of Γ_n intersecting $Q(C_1n)$. By the left inclusion in (24), this is less than or equal to the number of faces of Γ_∞ in $U_{\tau n}$. By preceding remarks, there is $C_2 > 0$ such that that number is bounded by $C_2 \text{area}(U_{\tau n})$. By the right inclusion in (24), $\text{area}(U_{\tau n})$ is quadratic in n . We have obtained the bound $f(n) = O(n^2)$.

Now for the lower bound. All regular points in X are periodic [13]. A face $F \subset X$ of Γ_k is *stable* if F is a face of Γ_∞ . Let $V_n \subset X$ be the set of points with period at most n . Each connected component of V_n is an open, stable face of Γ_n . By remarks above, the number of connected components of V_n has the same growth as the area of V_n , thus $\text{area}(V_n) = O(f(n))$. By Proposition 8 below, $\text{area}(V_n) \sim n^2$. This completes the proof of Theorem 6. \square

The following proposition was used in the proof of Theorem 6. It is also of independent interest.

Proposition 8. *Let P be a convex polygon and let $X_{\text{per}} \subset X$ be the set of periodic points of the outer billiard. For $x \in X_{\text{per}}$ let $p(x)$ be the period.*

1. *We have $|x| = O(p(x))$.*
2. *Let P be a rational polygon. Then for all regular points $p(x) \sim |x|$. Let V_n be the set of points such that $p(x) \leq n$. Then $\text{area}(V_n) \sim n^2$.*

Proof. We assume, without loss of generality, that $p(x) = 2m$. Let Q_x be the circle through point x . The sequence $x, T^2(x), \dots$ roughly follows Q_x . To come back to x , the sequence has to go around Q_x at least once. Let δ be the “largest step” of T^2 . Then we need at least $\text{perimeter}(Q_x)/\delta$ steps to return. Since $\text{perimeter}(Q_x) \sim |x|$, the first claim follows.

Let now P be a rational (hence quasirational) polygon, and let $U_k, k \geq 1$, be the Kolodziej domains. Let $k = k(x)$ be such that $x \in U_k \setminus U_{k-1}$. The relations $Q(Ck) \subset U_k \subset Q(C(k+1))$ imply that the function $k(x)$ satisfies $k(x) \sim |x|$. By inclusion $U_k \setminus U_{k-1} \subset Q(C(k+1)) \setminus Q(C(k-1))$, we have $\text{area}(U_k \setminus U_{k-1}) \sim |x|$. The point x belongs to a unique face, $F = F(x)$, of Γ_∞ , hence $p(x) \text{area}(F) \leq \text{area}(U_k \setminus U_{k-1})$. By preceding remarks, $p(x) = O(\text{area}(U_k \setminus U_{k-1}))$, implying $p(x) = O(|x|)$, and hence $p(x) \sim |x|$.

By this relation, there are constants $C_3, C_4 > 0$ such that for n sufficiently large, $Q(C_3n) \subset V_n \subset Q(C_4n)$, proving the last claim. \square

4.2. The elliptic and the hyperbolic cases. We will first consider the elliptic case.

Theorem 7. *Let $P \subset S^2$ be a convex spherical polygon. The complexity of the outer billiard about P is subexponential.*

Proof. For $x \in S^2$ let $l = x^*$ be the appropriately oriented great circle centered at x . This diffeomorphism $S^2 \rightarrow \text{Geo}(S^2)$ is the *spherical duality*, and we denote by $x = l^*$ the inverse diffeomorphism.

Let a, b, \dots be the corners of P , and let P^* be the convex polygon bounded by the geodesics a^*, b^*, \dots . The correspondence $P \mapsto P^*$ is an automorphism of the space of convex spherical polygons. The proof of the following lemma is contained in [16].

Lemma 10. *Let $P, P^* \subset S^2$ be as above. Let X_o, X_b be the phase spaces of the outer billiard about P , inner billiard in P^* ; let $T^{\text{out}}: X_o \rightarrow X_o, T_{\text{bil}}: X_b \rightarrow X_b$ be the respective maps.*

The spherical duality induces a diffeomorphism $X_o \rightarrow X_b$; it conjugates the mappings $T^{\text{out}}: X_o \rightarrow X_o$ and $T_{\text{bil}}: X_b \rightarrow X_b$; it induces an isomorphism of the codings.

Figure 9 illustrates Lemma 10. Let $f_o(n)$ (resp. $f_b(n)$) be the corner complexity of the outer billiard about P (resp. the side complexity of the billiard in P^*). By Lemma 10, $f_o(n) = f_b(n)$. The claim of Theorem 7 now follows from Theorem 3. □

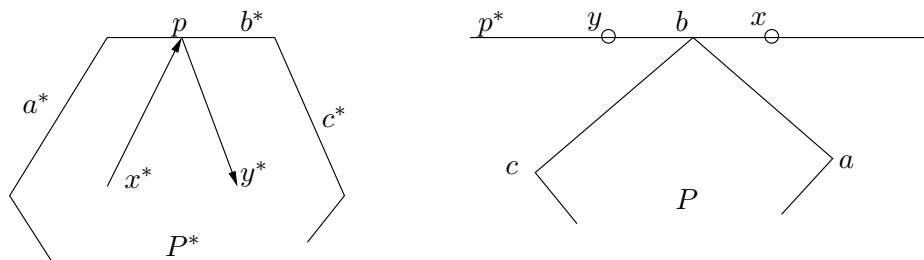


FIGURE 9. Duality between inner and outer billiards

Let $P \subset \mathbb{H}^2$ be a p -gon, and let $X = \mathbb{H}^2 \setminus P$. The outer billiard map $T: X \rightarrow X$ extends to a homeomorphism, $\tau: S \rightarrow S$, of the circle at infinity whose rotation number satisfies $\rho(P) \geq 1/p$ [8]. The polygon P is *large* if $\rho(P) = 1/p$ and τ has a hyperbolic p -periodic orbit. See figure 10. The set of large polygons is open in the natural topology [8].

Theorem 8. *Let $P \subset \mathbb{H}^2$ be an arbitrary convex polygon, and let $f(\cdot)$ be the complexity of the outer billiard. Then $n = O(f(n))$. If P is a large polygon, then $f(n) \sim n$.*

Proof. The bound $n = O(f(n))$ fails iff the sequence $\Gamma_k, k \geq 1$, stabilizes. Assume this to be the case, and let $\Gamma_m = \Gamma_{m+1} = \dots = \Gamma_\infty$. The outer billiard map, T , preserves Γ_∞ ; its restriction to a closed face of Γ_∞ is a diffeomorphism onto another one. Since Γ_∞ is a finite graph, we find $n \in \mathbb{N}$ such that every face of Γ_∞ is invariant under T^n .

Let F be a closed face of Γ_∞ . Then $\partial F \cap S$ is either empty, or a vertex, or an edge of F . We will study the latter. Let $v_1, \dots, v_N \in S$ be the consecutive endpoints

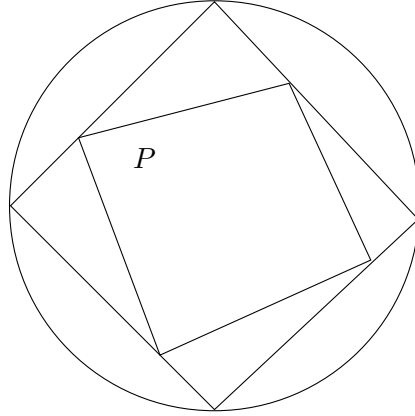


FIGURE 10. A large quadrilateral

of these edges, let $e_i \subset S$ (resp. $\alpha_i \subset \mathbb{H}^2$) be the circular arc (resp. the geodesic) with endpoints v_i, v_{i+1} (we set $N + 1 = 1$), and let F_i be the corresponding face of Γ_∞ . The restriction $T^n|_{F_i}$ is induced by an isometry, $g_i \in \text{Iso}(\mathbb{H}^2)$. The elements g_1, \dots, g_N are all equal to the identity iff $\tau^N = 1$.

Lemma 11. *The map $\tau: S \rightarrow S$ is not periodic.*

Proof. Let z be a corner of P . For close points $x_1, y_1 \in S$ let $x_2, y_2 \in S$ be their reflections about z . Let $\lambda_1 = |x_2z|/|x_1z|$ and let $2\alpha_i$ be the angular measure of the arc $x_iy_i, i = 1, 2$. See Fig. 11. The triangles x_1zy_1 and x_2zy_2 are similar, therefore

$$\sin \alpha_2 = \lambda_1 \sin \alpha_1. \tag{25}$$

Let x_1, \dots, x_N be a periodic trajectory of the map τ consisting of smooth points, and let $\lambda_1, \dots, \lambda_N$ be the respective ratios. Set $\Lambda = \prod_{i=1}^N \lambda_i$. Let y_1 be a point sufficiently close to x_1 , and let y_1, \dots, y_N be its τ -orbit; we assume that both orbits reflect in the same corners of P . It follows from equation (25) that y_1, \dots, y_N is a periodic trajectory iff $\Lambda = 1$. In particular, if τ has a periodic interval, then $\Lambda = 1$ there.

Let now x_1 cross counter-clockwise a singularity half-line of T . In the notation of figure 12, $\lambda_1 = (b + c)/a$ (resp. $\lambda_1 = c/(a + b)$) right before (resp. after) this. By $(b + c)/a > c/(a + b)$, the equality $\Lambda = 1$ before a singularity half-line implies that $\Lambda < 1$ immediately after it. (If several points simultaneously cross a singularity half-line, Λ will decrease as well.) \square

By Lemma 11, we can assume without loss of generality that $g_1 \neq 1$. Then g_1 is a (hyperbolic) parallel translation with the axis α_1 , and F_1 is the domain bounded by α_1 and e_1 . We will say that F_1 is a *lunar face* of Γ_∞ . The union of lunar faces of Γ_∞ is invariant under T . Therefore for any $k > 0$ there is $l = l(k)$ such that $T^{-k}(\alpha_1) = \alpha_l$. A geodesic $\alpha_i, 1 \leq i \leq N$, cannot contain a side of P . If it does,

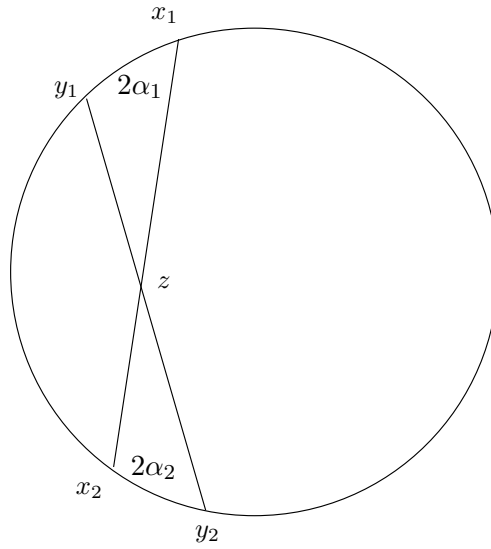


FIGURE 11. Computing the distortion of the map τ

then F_i contains a singular line of T in its interior, contrary to the definition of F_i . See Fig. 12, where x_1x_2 represents now the geodesic α_1 . Thus, α_1 is not an edge of Γ_m for any m . This contradiction proves our first claim.

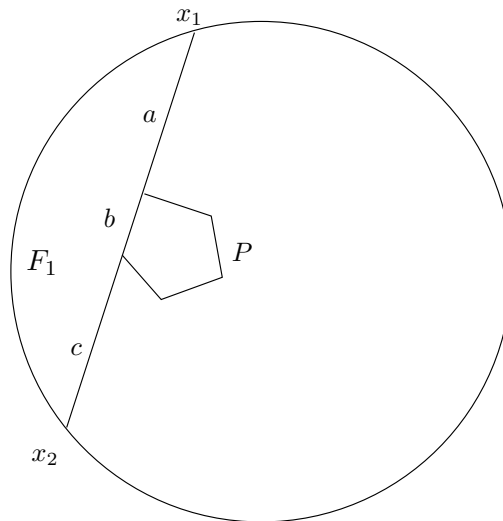
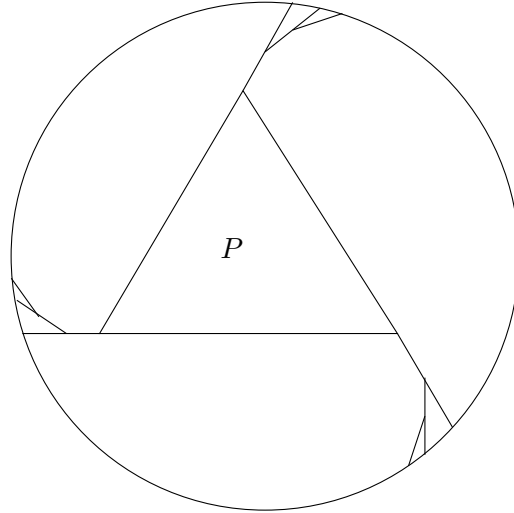


FIGURE 12. Destruction of a periodic orbit of τ

FIGURE 13. The graph Γ_2 for a large triangle

Let now P be a large p -gon. Then Γ_n is a disjoint union of p binary trees [8] (see Fig. 13), hence $|\Gamma_n|$ grows linearly. This completes the proof of Theorem 8. \square

Remark 3. The function $f(\cdot)$ is bounded below by the complexity of the induced map $\tau: S \rightarrow S$ with respect to the natural partition. However, the latter may be finite. See Fig. 14.

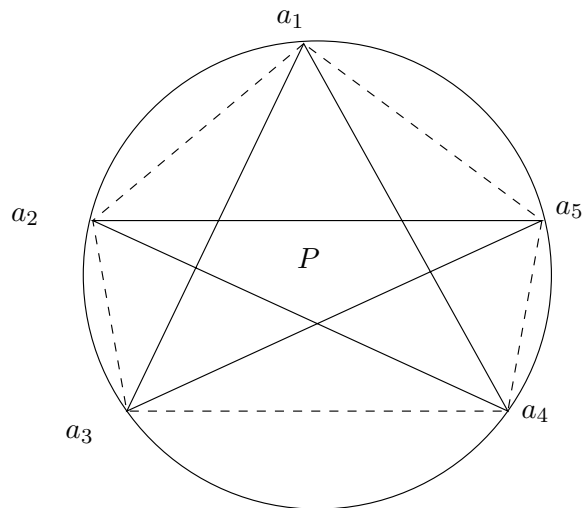


FIGURE 14. Finite complexity of the outer billiard map at infinity

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