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Commuting Dual Billiard Maps

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Abstract. To a closed convex smooth curve in the plane the dual billiard transformation of its exterior corresponds: given a point outside of the curve, draw a tangent line to it through the point, and reflect the point in the point of tangency. We prove that if two curves are given, such that the corresponding dual billiard transformations commute, then the curves are concentric homothetic ellipses.

Mathematics Subject Classification (1991): 70H99.

1. Introduction

The dual billiard map T is a map of the exterior of a smooth strictly convex closed plane curve γ into itself, which is defined as follows: given a point x outside of γ, draw the right (from the view-point of x) tangent line to γ through it and reflect x in the point of tangency to obtain the point $T(x)$ (see Figure 1). We refer to [T1] and [T2] for a discussion of various aspects of the dual billiard problem.

Observe that if $\gamma_1 \subset \gamma_2$ are two concentric circles, then the corresponding dual billiard maps in the exterior of γ_2 commute: $T_1T_2 = T_2T_1$ (see Figure 2). Since the dual billiard map commutes with affine transformations of the plane, the same commutation property holds when γ_1 and γ_2 are concentric homothetic ellipses. The aim of this note is to prove the converse statement.

THEOREM. *Let γ_1 and γ_2 be smooth strictly convex distinct closed plane curves, such that the corresponding dual billiard maps T_1 and T_2 commute: $T_1T_2 = T_2T_1$ in the domain, where the right- and the left-hand sides are defined. Then γ_1 and γ_2 are concentric homothetic ellipses.*

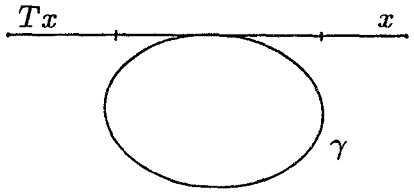


Fig. 1

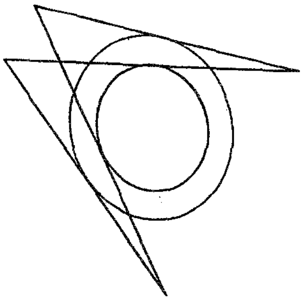
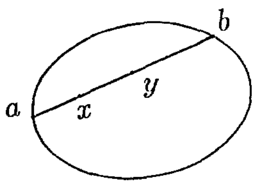


Fig. 2



$dist(x, y) = |log[x, y, a, b]|$

Fig. 3.

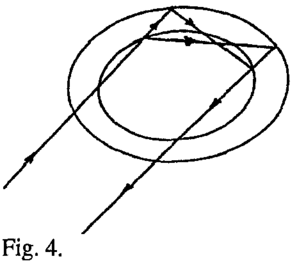


Fig. 4.

As a motivation to look at this problem, notice that the commutativity in question is closely related to integrability. Suppose a $2n$-dimensional symplectic manifold is foliated by Lagrangian tori. Then each torus carries a canonical affine structure; this structure is given by a locally free action of $\mathbb{R}^n$, defined by symplectic gradients of functions, constant on the leaves of the foliation. If two symplectic maps T_1 and T_2 preserve the foliation leaf-wise, then each is a shift in the affine structure of a leaf (see, e.g. [A-G]). Therefore, $T_1T_2 = T_2T_1$.

In our case, the symplectic manifold is the exterior of γ_2, foliated by circles, concentric to γ_1 and γ_2, and preserved by T_1 and T_2 (the dual billiard map is area-preserving for any curve γ – see the next section). Another example is provided by dual billiards in the hyperbolic plane (see [T3]). Let γ_∞ be an ellipse in the (flat) plane and consider its interior as the Klein's model of the hyperbolic plane (see Figure 3). If $\gamma_1 \subset \gamma_\infty$ is a convex curve, then the dual billiard map T_1 with respect to γ_1 (in the hyperbolic metric!) preserves the hyperbolic area. If γ_1 is an ellipse, then T_1 preserve ellipses, which belong to the pencil, generated by γ_1 and γ_∞, i.e. the one-parameter family of conics, passing through the four (complex) points of the intersection $\gamma_1 \cap \gamma_\infty$. This pencil constitute a Lagrangian foliation. One concludes, that if γ_2 is an ellipse from the pencil, then the corresponding dual billiard maps commute: $T_1T_2 = T_2T_1$.

Still another example is the billiard in an ellipsoid. The billiard transformation acts on the set of oriented lines (rays), sending a ray to the reflected one. The space of rays in $\mathbb{R}^{n+1}$ has a canonical symplectic structure (identifying it with T^*S^n) and the billiard transformation is symplectic. A quadric M in $\mathbb{R}^{n+1}$ gives rise to a Lagrangian foliation in the space of rays in $\mathbb{R}^{n+1}$, whose leaves consist of rays, tangent to n fixed quadrics, confocal to M. The billiard transformation with respect to M preserves the leaves of this foliation (see [A-G] for all these facts). We conclude, that if M_1 and M_2 are confocal ellipsoids, then the corresponding billiard transformations commute: $T_1T_2 = T_2T_1$ – see Figure 4 for the plane case. A similar result holds for quadrics in elliptic or hyperbolic spaces ([V1]).

We conjecture, that results, analogous to the above-stated theorem, hold true in all these examples (e.g. given two smooth closed convex plane curves, such that the corresponding billiard transformations commute in the domain, where both are defined, the curves are confocal ellipses). For a broad discussion on commuting

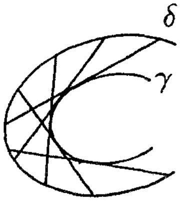


Fig. 5.

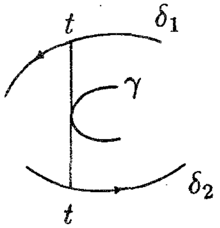


Fig. 6.

and integrable maps we refer to [V2]. The rest of this note is devoted to a proof of the theorem.

2. Proof

The proof is elementary and consists of several steps. We fix a linear structure in the plane and denote by $[\xi, \eta]$ the skew-symmetric bilinear form $\det(\xi, \eta)$.

STEP 1

This is an observation, made in [F-T]. It shows how to recover the dual billiard curve by an invariant curve of the dual billiard map. Its analog for billiards is known as Grave's theorem [B].

LEMMA 1.1. *Let δ be a closed convex curve. Consider a one-parameter family of lines, which cut off segments of fixed areas from δ, and let γ be its envelope. Then δ is invariant under the dual billiard map with respect to γ* (Figure 5). *Singular points of γ correspond to the lines, which meet δ at points with parallel tangent lines.*

Proof. Let t be a variable parametrizing the family of lines. Then t is a local parameter on pieces of δ near the points of intersection with the line (Figure 6), and also a parameter on γ. Denote d/dt by a dot, and let $A(t)$ be the area, cut off by the line. Then

$$0 = \dot{A}(t) = \tfrac{1}{2}[\delta_2(t) - \delta_1(t),\ \dot{\delta}_1(t) + \dot{\delta}_2(t)]. \tag{1}$$

It follows, that the mid-point of the segment $(\delta_1(t),\ \delta_2(t))$ moves along this segment, i.e. $\frac{1}{2}(\delta_1(t) + \delta_2(t)) \in \gamma$. Whence the first claim.

If $\dot{\delta}_1(t)$ and $\dot{\delta}_2(t)$ are not parallel, then $\dot{\gamma}(t) = \frac{1}{2}(\dot{\delta}_1(t) + \dot{\delta}_2(t)) \neq 0$. If $\dot{\delta}_1(t)$ and $\dot{\delta}_2(t)$ are parallel, then (1) can hold only if $\dot{\delta}_1(t) = -\dot{\delta}_2(t)$. This is a singular point of the envelope: $\dot{\gamma}(t) = 0$. □

COROLLARY 1.2. *The dual billiard map is area-preserving.*

Proof. Shaded areas in Figure 7 are equal. □

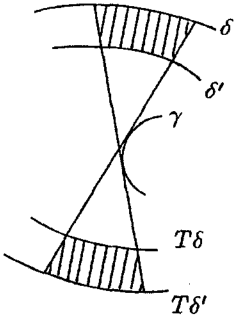


Fig. 7.

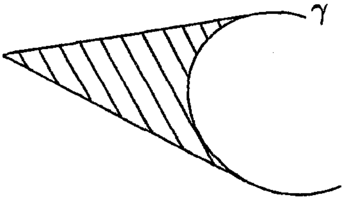


Fig. 8

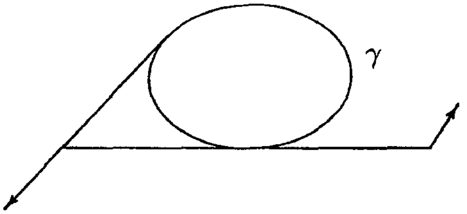


Fig. 9.

If one identifies the exterior of γ with a semi-infinite cylinder and takes tangent half-lines to γ as a 'vertical' foliation, it is easy to find the generating function for T: it is equal to the shaded area in Figure 8 ([T1], [T2]). We will not use this fact here. Also T satisfies the twist condition, i.e. it rotates 'vertical' vectors in the same sense (see Figure 9).

STEP 2

Let γ be a smooth closed strictly convex dual billiard curve and let T be the corresponding dual billiard map. Parametrize γ by a parameter t, so that $[\dot\gamma(t), \ddot\gamma(t)] = 1$. Differentiating, we find $[\dot\gamma, \dddot\gamma] = 0$; hence $\dddot\gamma(t) = -K(t)\dot\gamma(t)$ for a function $K(t)$. The parameter t is called affine-length parameter, and $K(t)$ is affine curvature of γ (both are invariant under area-preserving affine transformations of the plane). Each point (x, y) outside of γ can be represented as $\gamma(t) + r\dot\gamma(t)$ with $r \geq 0$.

LEMMA 2.1. *The area-form* $\mathrm{d}x \wedge \mathrm{d}y = r\,\mathrm{d}r \wedge \mathrm{d}t$.

Proof. We have

$$x = \gamma_1(t) + r\dot\gamma_1(t), \qquad y = \gamma_2(t) + r\dot\gamma_2(t),$$

where $\gamma_{1,2}$ are components of the vector γ. It follows that

$$\mathrm{d}x \wedge \mathrm{d}y = r[\dot\gamma, \ddot\gamma]\,\mathrm{d}r \wedge \mathrm{d}t = r\,\mathrm{d}r \wedge \mathrm{d}t. \qquad \square$$

We will need explicit formulas for T in a vicinity of γ.

LEMMA 2.2. *Let* $T(t, r) = (t_1, r_1)$. *Then*

$$t_1 = t + 2r - \tfrac{2}{3}K(t)r^3 - \tfrac{8}{15}\dot{K}(t)r^4 + O(r^5);$$

$$r_1 = r + \tfrac{2}{15}\dot{K}(t)r^4 + O(r^5). \qquad \square$$

These formulas are derived from the equation

$$\gamma(t) + r\dot{\gamma}(t) = \gamma(t_1) - r_1\dot{\gamma}(t_1)$$

by a direct computation, which we omit.

STEP 3

We apply the KAM theory to establish existence of invariant curves of the dual billiard map in an arbitrary small neighbourhood of the dual billiard curve, as well as arbitrarily far away from it.

LEMMA 3.1. *For any* $\varepsilon > 0$ *there exists a* T*-invariant curve* δ *in the* ε*-neighbourhood of* γ*, and outside of the* $1/\varepsilon$*-neighbourhood of* γ*, such that* T*, restricted to* δ*, is conjugated to a rotation of a circle over an angle, incommensurable with* π.

Proof. It follows from the formulas of Lemma 2.2 that T is a small perturbation of the map $T_0(t, r) = (t + 2r, r)$ near γ. The map T_0 is integrable, the curves r-const being T_0-invariant. The KAM theory implies that most of these curves, whose rotation numbers are irrational and satisfy certain Diophantine conditions, persist under a small perturbation. Whence the claim about a small neighbourhood of γ.

As to a neighbourhood of infinity, one considers the square map T^2 instead of T. It was remarked in [M1], [M2] that this map is a small perturbation of an integrable map far away from γ. We refer to [T1], [T2] and [M-T] for details. It follows again that T^2 possesses invariant curves. Let δ be a curve such that the T^2-orbit of each of its point is dense in δ. Then $\delta_1 = T\delta$ is also T^2-invariant. Since T is area-preserving, δ and δ_1 intersect. If $x \in \delta \cap \delta_1$, then the closure of the T^2-orbit of x is δ. On the other hand, it belongs to δ_1. Hence $\delta_1 = \delta$, so δ is T-invariant. $\square$

Let γ_1 and γ_2 be two curves and let T_1 and T_2 be the corresponding dual billiard maps, which commute in the domain, where both maps T_1T_2 and T_2T_1 are defined.

LEMMA 3.2. *One of the curves lies strictly inside the other.*

Proof. The maps T_1T_2 and T_2T_1 are defined outside of a sufficiently big circle, containing γ_1 and γ_2. Let δ be a T_1-invariant curve in the exterior of this circle, satisfying the condition of the previous lemma. Since $T_1T_2 = T_2T_1$, the curve $T_2\delta$ is also T_1-invariant. Repeating the argument from the previous proof, one concludes that δ and $T_2\delta$ intersect and, hence, $T_2\delta = \delta$.

Since δ is invariant under T_1 and T_2, we apply Lemma 1.1 to recover γ_1 and γ_2. That is, there exist positive reals $a_1 \neq a_2$, which are less than the area, bounded by δ, such that γ_i is the envelope of lines cutting off the area of a_i from δ $(i = 1,\ 2)$. If neither curve contains the other, then there exists a common tangent line of γ_1 and γ_2. This line cuts off the area from δ, equal to a_1 and a_2 at the same time, which contradicts $a_1 \neq a_2$. One easily sees, that $\gamma_2 \subset \gamma_1$ if $a_1 < a_2$, and $\gamma_1 \subset \gamma_2$ if $a_2 < a_1$. □

STEP 4

Assume that $\gamma_2 \subset \gamma_1$ are two dual billiard curves and $T_1T_2 = T_2T_1$. From now on we consider a small neighbourhood of γ_1. We parametrize γ_1 by the affine parameter t and, applying a dilation, if necessary assume that its affine length equals 1, i.e. $t \in [0,\ 1]$. It follows from the previous step that there is an abundance of curves invariant under T_1 and T_2 near γ_1. Choosing a sequence of such curves, which tends to γ_1, one concludes that $T_2(\gamma_1) = \gamma_1$. In particular, the domain where $T_1T_2 = T_2T_1$ is the exterior of γ_1.

One would like to parametrize the set of invariant curves near γ_1. A natural parameter is the rotation number of T_1 on such a curve, as follows from the twist property of T_1. This parameter takes values in a Cantor set, which has zero as an accumulation point. We choose, however, another parameter. If δ is a T_1-invariant curve near γ_1, define

$$\varepsilon(\delta) = (2A(\delta))^{1/2} - \frac{1}{15}\left(\int_0^1 K(t)\,\mathrm{d}t\right)(2A(\delta))^{3/2},$$

where $A(\delta)$ is the area between δ and γ_1 and $K(t)$ is the affine curvature of γ_1. The parameter ε also takes values in a certain Cantor set E, containing zero as an accumulation point. We write δ_ε to express the fact that $\varepsilon(\delta) = \varepsilon$.

LEMMA 4.1. *In the coordinates $(t,\ r)$ near γ_1, the curve δ_ε has the equation*

$$r = \varepsilon + \tfrac{1}{15}K(t)\varepsilon^3 + f(t,\ \varepsilon)\varepsilon^5, \tag{2}$$

where $f(t,\ \varepsilon)$ is some function.

Proof. Consider a curve δ given by equation (2). First, we find $A(\delta)$. According to Lemma 2.1 and (2),

$$(2A(\delta))^{1/2} = \left(\int_0^1 r^2\,\mathrm{d}t\right)^{1/2} = \varepsilon + \frac{\varepsilon^3}{15}\int_0^1 K(t)\,\mathrm{d}t + O(\varepsilon^5).$$

It follows, that $\varepsilon(\delta) = \varepsilon + O(\varepsilon^5)$.

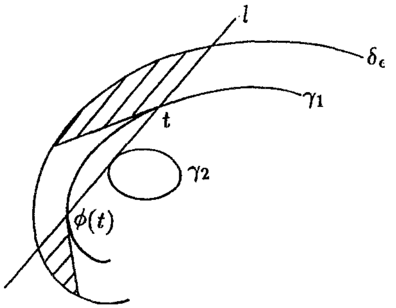


Fig. 10.

Suppose now that $T_1(t, r) = (t_1, r_1)$, where (t, r) satisfy (2). One needs to check that

$$r_1 = \varepsilon + \tfrac{1}{15}K(t_1)\varepsilon^3 + O(\varepsilon^5). \tag{3}$$

By Lemma 2.2 and (2),

$$r_1 = r + \tfrac{2}{15}\dot{K}(t)r^4 + O(r^5) = \varepsilon + \tfrac{1}{15}K(t)\varepsilon^3 + \tfrac{2}{15}\dot{K}(t)\varepsilon^4 + O(\varepsilon^5).$$

By the same formulas, the right-hand side of (3) equals

$$\varepsilon + \tfrac{1}{15}K(t+2r)\varepsilon^3 + O(\varepsilon^5) = \varepsilon + \tfrac{1}{15}K(t)\varepsilon^3 + \tfrac{2}{15}\dot{K}(t)\varepsilon^4 + O(\varepsilon^5).$$

Hence, equality (3) holds. □

STEP 5

Consider the map $T_2{:}\gamma_1 \to \gamma_1$, where γ_1 is parametrized by the affine length t. Then $T_2(\gamma_1(t)) = \gamma_1(\varphi(t))$ for a homeomorphism of the circle $\varphi(t)$.

LEMMA 5.1. *There exists a constant a such that $\varphi(t) = t + a$.*

Proof. Join the points $\gamma_1(t)$ and $\gamma_1(\varphi(t))$ by a line l. Consider a curve δ_ε, invariant under T_1 and T_2 (Figure 10). By Lemma 1.1, the area of the domain bounded by l and γ_1 does not depend on t. The same holds true for δ_ε. Hence, the area A of the strip between γ_1 and δ_ε, shown in Figure 11, is independent of t. We will compute A modulo terms of order ε^3.

Draw tangent half-lines to γ_1 through the points $\gamma_1(t)$ and $\gamma_1(\varphi(t))$. We obtain two triangles, shaded in Figure 10. Their areas are of order ε^3. We replace the strip in Figure 11 by the one in Figure 12. By Lemma 2.1, its area equals $\frac{1}{2}\int_t^{\varphi(t)} r^2(\tau)\,d\tau$, where $r = r(t)$ is the equation of δ_ε in (t, r)-coordinates. By Lemma 4.1,

$$A = \frac{\varepsilon^2}{2}(\varphi(t) - t) + O(\varepsilon^3).$$

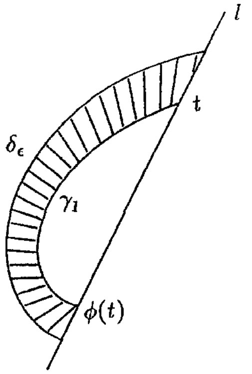


Fig. 11.

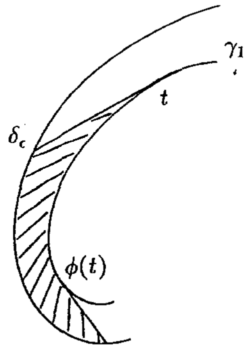


Fig. 12.

Divide by ε^2 and take the limit $\varepsilon \to 0$ over the Cantor set E, in which ε takes values. Since A is independent of t, one concludes that $\varphi(t) - t = \text{const}$. □

In the next section we will evaluate the area A up to the terms of order ε^4, to conclude that the affine curvature $K(t)$ of the curve γ_1 is constant. To this end we need another formula. Let a be as in the previous lemma. Denote $\dot\gamma_1(t)$ by u and $\dot\gamma_1(t+a)$ by v. By Lemma 1.1, there exists a non-vanishing functions $f(t)$ such that

$$u + v = f(t)(\gamma_1(t+a) - \gamma_1(t)). \tag{4}$$

LEMMA 5.2. *There exists a constant c such that*:

$$\dot u = \left(\frac{\dot f}{f} - f + \frac{c}{f^2}\right) u + \frac{c}{f^2} v; \qquad \dot v = -\frac{c}{f^2} u + \left(\frac{\dot f}{f} + f - \frac{c}{f^2}\right) v; \tag{5}$$

$$K(t) = 3\dot f - \frac{\ddot f}{f} - f^2 + \frac{2c}{f}, \qquad K(t+a) = -3\dot f - \frac{\ddot f}{f} - f^2 + \frac{2c}{f}. \tag{6}$$

Here $K(t)$ is the affine curvature of $\gamma_1(t)$; the dot is d/d*t.*

Proof. Write

$$\dot u = pu + qv, \qquad \dot v = \bar p u + \bar q v,$$

where $p,\ q,\ \bar p,\ \bar q$ are unknown functions. Divide (4) by f and differentiate to find

$$p + \bar p = \frac{\dot f}{f} - f, \qquad q + \bar q = \frac{\dot f}{f} + f. \tag{7}$$

Since $[u,\ \dot u] = [v,\ \dot v] = 1$, one obtains

$$q = \frac{1}{[u,\ v]}, \qquad \bar p = -\frac{1}{[u,\ v]}. \tag{8}$$

Hence, $p+\bar{q}=2\dot{f}/f$. On the other hand,

$$[u,\, v]^{\cdot}=[\dot{u},\, v]+[u,\, \dot{v}]=\frac{2\dot{f}}{f}[u,\, v].$$

Therefore, $[u,\, v]=f^2/c$ for a constant c. We find q and $\bar{p}$ from (8), and p and $\bar{q}$ from (7). Formula (5) follows. To obtain (6), differentiate (5) and use the definition of curvature: $\ddot{u}=-Ku$. □

Notice, that (6) implies a certain difference-differential equation for the function $f(t)$. If one could deduce that $f(t)=\text{const}$, then $K(t)$ would be constant and γ_1 an ellipse. However, this equation is too complicated to deal with, and we proceed in a different way.

STEP 6 (and the last)

We evaluate the area $A(t)$ of the strip shaded in Figure 11 up to the terms of order ε^4. Let $(t-\beta_1,\, r_1)$ and $(t+a-\beta_2,\, r_2)$ be $(t,\, r)$-coordinates of the points of intersection of l with δ_ε.

LEMMA 6.1.

$$\beta_1=\varepsilon-\frac{1}{2}\left(\frac{\dot{f}(t)}{f(t)}-f(t)\right)\varepsilon^2+O(\varepsilon^3),$$

$$\beta_2=\varepsilon-\frac{1}{2}\left(\frac{\dot{f}(t)}{f(t)}+f(t)\right)\varepsilon^2+O(\varepsilon^3).$$

Proof. Let $\beta_1=p\varepsilon+q\varepsilon^2+O(\varepsilon^3)$. Then

$$\gamma_1(t-\beta_1)-\gamma_1(t)+r(t-\beta_1)\dot{\gamma}_1(t-\beta_1) \tag{9}$$

is collinear with $u+v$; here $r(t)$ is given by (2). Consider a Taylor expansion of (9) at t to replace it by

$$\varepsilon(1-p)u-\varepsilon^2\left(qu+p\dot{u}-\frac{p^2}{2}\dot{u}\right)+O(\varepsilon^3). \tag{10}$$

Since (10) is collinear with $u+v$, we conclude that $p=1$. Substitute $\dot{u}$ from (5) to replace (10) by

$$-\varepsilon^2\left(qu+\frac{1}{2}\left(\frac{\dot{f}}{f}-f+\frac{c}{f^2}\right)u+\frac{c}{2f^2}v\right)+O(\varepsilon^3).$$

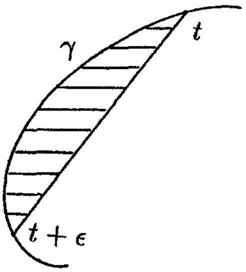


Fig. 13.

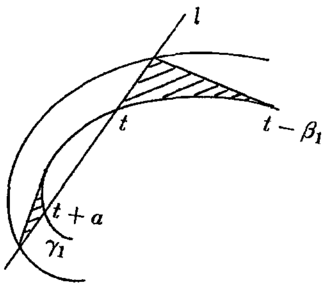


Fig. 14.

Since it is still collinear with $u+v$, we find

$$q=-\frac{1}{2}\left(\frac{\dot{f}}{f}-f\right).$$

Similarly one finds β_2. □

Let $\gamma(t)$ be a convex curve, parametrized by the affine parameter and let $A(t,\,\varepsilon)$ be the area, bounded by γ and the line, joining the points $\gamma(t)$ and $\gamma(t+\varepsilon)$ (Figure 13).

LEMMA 6.2. $A(t,\,\varepsilon)=\frac{1}{12}\varepsilon^3+O(\varepsilon^5)$.

Proof. Evidently,

$$\frac{\partial A(t,\,\varepsilon)}{\partial\varepsilon}=\frac{1}{2}[\gamma(t+\varepsilon)-\gamma(t),\,\dot{\gamma}(t+\varepsilon)].$$

Using Taylor expansions of $\gamma(t+\varepsilon)$ and $\dot{\gamma}(t+\varepsilon)$ at t, we get

$$\frac{\partial A}{\partial\varepsilon}=\frac{1}{2}\left[\varepsilon\dot{\gamma}(t)+\frac{\varepsilon^2}{2}\ddot{\gamma}(t)+\frac{\varepsilon^3}{6}\dddot{\gamma}(t),\,\dot{\gamma}(t)+\varepsilon\ddot{\gamma}(t)+\frac{\varepsilon^2}{2}\dddot{\gamma}(t)\right]+O(\varepsilon^4).$$

Since $[\dot{\gamma},\,\ddot{\gamma}]=1$ and $[\dot{\gamma},\,\dddot{\gamma}]=0$, this amounts to

$$\frac{\partial A}{\partial\varepsilon}=\frac{\varepsilon^2}{4}+O(\varepsilon^4).$$

Hence, $A=(\varepsilon^3/12)+O(\varepsilon^5)$. □

Denote by S_1 and S_2 the shaded areas in Figure 14. Then

$$A(t)=\frac{1}{2}\int_{t-\beta_1}^{t+a-\beta_2}r^2(\tau)\,\mathrm{d}\tau-S_1+S_2, \tag{11}$$

where, as before,

$$r(t)=\varepsilon+\tfrac{1}{15}K(t)\varepsilon^3+O(\varepsilon^5). \tag{12}$$

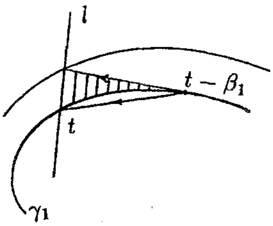


Fig. 15.

LEMMA 6.3. *The affine curvature $K(t)$ of the curve γ_1 is constant.*

Proof. First we find S_1 and S_2. According to Figure 15 and the previous lemma,

$$S_1 = \tfrac{1}{2}[\gamma_1(t-\beta_1) - \gamma_1(t),\ r(t-\beta_1)\dot\gamma_1(t-\beta_1)] - \tfrac{1}{12}\beta_1^3 + O(\varepsilon^5).$$

Here $\beta_1 = \varepsilon - p\varepsilon^2 + O(\varepsilon^3)$ with $p = \frac{1}{2}((\dot f/f) - f)$ by Lemma 6.1. As in the previous proof, we take Taylor expansions, using that $[u,\ \dot u] = 1$ and $[u,\ \ddot u] = 0$:

$$\begin{aligned} S_1 &= \frac{1}{2}\left[-\beta_1 u + \frac{\beta_1^2}{2}\dot u - \frac{\beta_1^3}{6}\ddot u,\ \left(\varepsilon + \frac{1}{15}K(t)\varepsilon^3\right)\left(u - \beta_1 \dot u + \frac{\beta_1^2}{2}\ddot u\right)\right] \\ &\quad -\frac{1}{12}\beta_1^3 + O(\varepsilon^5) \\ &= \frac{\beta_1\varepsilon^2}{4}(1-\varepsilon p) - \frac{1}{12}\beta_1^3 + O(\varepsilon^5) = \frac{\varepsilon^3}{6} - \frac{p\varepsilon^4}{4} + O(\varepsilon^5). \end{aligned}$$

Similarly, $\beta_2 = \varepsilon - q\varepsilon^2 + O(\varepsilon^3)$, where $q = \frac{1}{2}((\dot f/f) + f)$. Hence

$$S_2 = \frac{\varepsilon^3}{6} - \frac{q\varepsilon^4}{4} + O(\varepsilon^5).$$

Therefore, $S_1 - S_2 = \frac{1}{4}f(t)\varepsilon^4 + O(\varepsilon^5)$.

Next, we evaluate

$$I := \int_{t-\beta_1}^{t} r^2(\tau)\,\mathrm{d}\tau - \int_{t+a-\beta_2}^{t+a} r^2(\tau)\,\mathrm{d}\tau.$$

By (12) and Lemma 6.1,

$$I = (\beta_1 - \beta_2)\epsilon^2 + O(\varepsilon^5) = f(t)\varepsilon^4 + O(\varepsilon^5).$$

Hence, by (11)

$$A(t) = \frac{1}{2}\int_t^{t+a} r^2(\tau)\,\mathrm{d}\tau + \tfrac{1}{2}I - S_1 + S_2 = \frac{1}{2}\int_t^{t+a} r^2(\tau)\,\mathrm{d}\tau + \tfrac{1}{4}f(t)\varepsilon^4 + O(\varepsilon^5$$

Differentiating, one obtains in view of (12)

$$\begin{aligned} 0 = \dot{A}(t) &= \tfrac{1}{2}(r^2(t+a) - r^2(t)) + \tfrac{1}{4}\dot{f}(t)\varepsilon^4 + O(\varepsilon^5) \\ &= (\tfrac{1}{15}(K(t+a) - K(t)) + \tfrac{1}{4}\dot{f}(t))\varepsilon^4 + O(\varepsilon^5). \end{aligned}$$

By (6), $K(t+a) - K(t) = -6\dot{f}(t)$, and we conclude

$$0 = (\tfrac{1}{4} - \tfrac{6}{15})\dot{f}(t)\varepsilon^4 + O(\varepsilon^5).$$

Therefore, $\dot{f}(t) = 0$. It follows from (6), that $K(t) = \text{const}$. □

Curves, whose affine curvature is constant, are conics ([S]). Hence γ_1 is an ellipse. Apply a linear transformation to make it into a circle. Recover γ_2 by Lemma 1.1 to see that it is a concentric circle. This completes the proof of the theorem.

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