# Non-existence of $n$-dimensional $T$-embedded discs in $\mathbf{R}^{2n}$

Gordana Stojanovic and Serge Tabachnikov*
*Department of Mathematics, Penn State University*
*University Park, PA 16802, USA*

**Abstract**. We prove non-existence of $C^2$-smooth embeddings of $n$-dimensional discs to $\mathbf{R}^{2n}$ such that the tangent spaces at distinct points are pairwise disjoint.

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A number of recent papers concerned various non-degeneracy conditions on embedding and immersions of smooth manifolds in affine and projective spaces defined in terms of mutual positions of the tangent spaces at distinct points, see [1, 2, 3, 4, 7, 8, 9, 10, 11]. Following Ghomi [1], a $C^1$-embedded manifold $M^n \subset \mathbf{R}^N$ is called $T$-embedded if the tangent spaces to $M$ at distinct points do not intersect. For example, the cubic curve $(x, x^2, x^3)$ is a $T$-embedding of $\mathbf{R}$ to $\mathbf{R}^3$, and the direct product of such curves gives a $T$-embedding of $\mathbf{R}^n$ to $\mathbf{R}^{3n}$.

A $T$-embedding $M^n \to \mathbf{R}^N$ induces a topological embedding of the tangent bundle $TM \to \mathbf{R}^N$, hence $N \geq 2n$. One of the results in [1] is that no closed manifold $M^n$ admits $T$-embeddings to $\mathbf{R}^{2n}$. In this note we extends this result as follows (note that we assume more differentiability than Ghomi).

**Theorem 1** *There exist no $C^2$-smooth $T$-embedded discs $D^n$ in $\mathbf{R}^{2n}$.*

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**Proof**. Arguing by contradiction, assume that such a disc $D^n$ exists. Choose the tangent space at the origin and its orthogonal complement as coordinate $n$-dimensional spaces. Making $D$ smaller, if necessary, assume that the disc is the graph of a (germ of a) $C^2$ smooth map $f : \mathbf{R}^n \to \mathbf{R}^n$. Let $U \subset \mathbf{R}^n$ be the domain of $f$.

Let $z = (u, f(u)) \in D$ where $u \in U$. The tangent space $T_z D$ is given by a linear equation $y = A(u)x - b(u)$ where $A(u)$ is an $n \times n$ matrix and $b(u)$ is a vector in $\mathbf{R}^n$, both depending on $u$. In terms of $f$, they have the following expressions. Let $f_1, ..., f_n$ be the components of $f$.

**Lemma 1.1** *One has:*

$$A_{ij} = \frac{\partial f_i}{\partial u_j}, \quad b_i = \sum_{k=1}^{n} \frac{\partial f_i}{\partial u_k} u_k - f_i.$$

**Proof**. The first statement is obvious, and the second follows from the fact that the space $y = A(u)x - b(u)$ passes through the point $z = (u, f(u))$. $\Box$

One has the next characterization of $t$-discs.

**Lemma 1.2** *For all $u \neq v \in U$, the vector $b(u) - b(v)$ does not belong to* $\mathrm{Im}(A(u) - A(v))$.

**Proof**. The spaces $y = A(u)x - b(u)$ and $y = A(v)x - b(v)$ intersect if and only if $b(u) - b(v) \in \mathrm{Im}(A(u) - A(v))$. $\Box$

**Lemma 1.3** *If $u \neq v$ then $b(u) \neq b(v)$ and $A(u) - A(v)$ is degenerate.*

**Proof**. The first claim follows from the fact that zero vector lies in any subspace, contradicting Lemma 1.2. If $A(u) - A(v)$ is nondegenerate then it is surjective, again contradicting Lemma 1.2. $\Box$

Now we compute the Jacobian of the map $b : U \to \mathbf{R}^n$. Denote by $E$ the Euler vector field in $\mathbf{R}^n$:

$$E = \sum_{k=1}^{n} u_k \frac{\partial}{\partial u_k}.$$

**Lemma 1.4** *One has:*

$$\frac{\partial b_i}{\partial u_j} = \sum_k \frac{\partial^2 f_i}{\partial u_j \partial u_k} u_k = E(A_{ij}).$$

**Proof**. This follows from Lemma 1.1. □

**Lemma 1.5** *For all $u \in U$, the Jacobian $Jb$ of the map $b$ is degenerate.*

**Proof**. Lemma 1.4 implies that

$$Jb = \lim_{\varepsilon \to 0} \frac{A(u + \varepsilon u) - A(u)}{\varepsilon}.$$

By Lemma 1.3 with $v = u + \varepsilon u$, the numerator is a degenerate matrix for all $\varepsilon$, and so is its quotient by $\varepsilon$. Thus $Jb$ is a limit of degenerate matrices. Since determinant is a continuous function, the limit also has zero determinant and therefore is degenerate. □

Finally, we arrive at a contradiction. By Lemma 1.3, the map $b$ is one-to-one, and by the invariance of domain theorem, its image has positive measure. By Lemma 1.5, every value of $b$ is singular, and by Sard's Lemma its image has zero measure. This completes the proof of Theorem 1. □

According to Lemma 1.3, the $n$-parameter family of $n \times n$ matrices $A(u), u \in D^n$ enjoys the property that $A(u) - A(v)$ is degenerate for all $u \neq v$. If $n = 2$, such families can be explicitly described. Assume that not all matrices $A(u)$ are zero.

**Theorem 2** *The family $A(u)$ consists either of the matrices with a fixed 1-dimensional image or with a fixed 1-dimensional kernel.*

**Proof**. Let $M_2$ be the space of linear maps $\mathbf{R}^2 \to \mathbf{R}^2$. One has a non-degenerate quadratic form in $M_2$ given by the determinant of a matrix; this form has signature $(2, 2)$. Consider the respective dot product.

Let $V \subset M_2$ be the linear span of the family $A(u)$.

**Lemma 2.1** *The subspace $V$ is isotropic.*

**Proof**. It suffices to prove that $A(u) \cdot A(v) = 0$ for all $u, v$. If $u = v$, this means precisely that $A(u)$ is degenerate. For $u \neq v$, the matrix $A(u) - A(v)$ is degenerate, hence $(A(u) - A(v)) \cdot (A(u) - A(v)) = 0$. Using bilinearity of the dot product, it follows that $A(u) \cdot A(v) = 0$. □

Since the dot product is non-degenerate, an isotropic subspace is at most 2-dimensional.

**Lemma 2.2** *A 2-dimensional isotropic subspace in* $M_2$ *consists either of the matrices with a fixed* 1*-dimensional image or with a fixed* 1*-dimensional kernel.*

**Proof**. Let $A \in V$ be a non-zero matrix. Choose a basis in the target space $\mathbf{R}^2$ in such a way that Im $A$ is orthogonal to the column vector $(0,1)$. Then

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

with $a^2+b^2 \neq 0$. Let $B \in V$ be another matrix, not proportional to $A$. Then $A \cdot B = 0$, and hence

$$B = \begin{pmatrix} c & d \\ at & bt \end{pmatrix}$$

for some real $c, d, t$. If $t = 0$ then $(c, d)$ is not proportional to $(a, b)$, and the space $V$ consists of matrices with zero second row. This is the first case of the lemma: the matrices have a fixed image spanned by the column vector $(1, 0)$.

Otherwise, $t \neq 0$. Since $B$ is degenerate, one has: $(c, d) = s(a, b)$ for some real $s$. Then

$$\frac{B - sA}{t} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix},$$

and the space $V$ consists of matrices with a fixed kernel spanned by the column vector $(-b, a)$. □

Lemma 2.2 obviously implies Theorem 2. □

For $n = 2$, Theorem 2 implies the claim of Theorem 1. Indeed, assume that the Jacobi matrix $Jf$ has a fixed 1-dimensional kernel, say, spanned by vector $\xi$. Then the map $f$ has zero directional derivative along $\xi$, and the tangent planes to the graph of $f$ are the same along this direction. Hence this graph is not $T$-embedded. Likewise, if $Jf$ has a fixed 1-dimensional image then the transpose matrix has a fixed kernel, say, $\eta$. This implies that the function $f(u) \cdot \eta$ has zero differential, and hence the image of $f$ is 1-dimensional. It follows that the graph of $f$ belongs to a 3-dimensional space and therefore is not $T$-embedded.

Let us conclude with two examples motivated by the following erroneous attempt to prove Theorem 1: if there exists a $T$-embedded disc $D^n \subset \mathbf{R}^{2n}$

then its tangent spaces provide a foliation $\mathcal{F}$ of a domain in $\mathbf{R}^{2n}$ by $n$-dimensional affine subspaces. Then $D^n$ is everywhere tangent to the leaves of this $n$-dimensional foliation and therefore must lie within a leaf. The mistake in this argument is that, no matter how smooth the embedding is, the foliation $\mathcal{F}$ may be not differentiable. This phenomenon is illustrated in the following example.

**Example 1** Let $\gamma$ be a smooth plane curve with positive curvature and free from vertices (extrema of curvature). Then, by the classical Kneser theorem (1912), the osculating circles to $\gamma$ are pairwise disjoint and nested as illustrated in figure 1; see, e.g., [6]. These osculating circles foliate the annulus $A$ between the largest and smallest of them. Denote this foliation by $\mathcal{F}$. Then $\mathcal{F}$ is not $C^1$, namely, one has the following result.

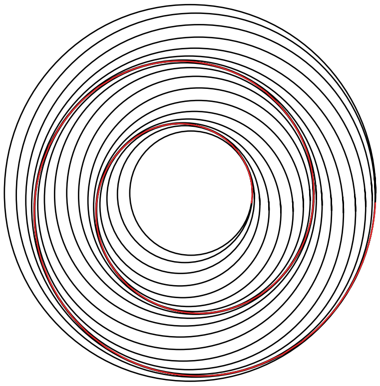

Figure 1: Osculating circles of a spiral

**Proposition 1.1** *Let $f : A \to \mathbf{R}$ be a differentiable function, constant on the leaves of $\mathcal{F}$. Then $f$ is constant in $A$.*

**Proof**. Since $f$ is constant on the leaves of $\mathcal{F}$, the differential $df$ vanishes on any vector tangent to any leaf. Since $\gamma$ is everywhere tangent to the leaves, $df$ is zero on the tangent vectors to $\gamma$. Hence $f$ is constant on $\gamma$. But $A$ is the union of the leaves of $\mathcal{F}$ through the points of $\gamma$, hence $f$ is constant in $A$. $\Box$

One also wonders whether $\mathbf{R}^{2n}$ can be foliated by non-parallel affine $n$-dimensional subspaces (clearly impossible for $n = 1$).

**Example 2** The following construction gives a foliation of $\mathbf{R}^4$ by pairwise non-parallel 2-dimensional affine subspaces. Start with partitioning 3-dimensional space into the vertical $z$-axis and the hyperboloids of 1 sheet

$$x^2 + y^2 = t(z^2 + 1), \quad t > 0$$

(when $t = 0$, one has the $z$-axis). Each hyperboloid is foliated by lines, and thus $\mathbf{R}^3$ gets foliated by lines; these lines are pairwise skew. Multiply this foliation by $\mathbf{R}^1$ to obtain the desired example.

This example, of course, is the Hopf fibration of 3-dimensional sphere by great circles, "in disguise": the radial projection of the sphere on $\mathbf{R}^3$ yields a foliation of space by pairwise skew lines. For classification of foliations of $S^3$ by great circles see [5].

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