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Periodic trajectories in 3-dimensional convex billiards

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Abstract. We give a lower bound on the number of periodic billiard trajectories inside a generic smooth strictly convex closed surface in 3-space: for odd n, there are at least $2(n-1)$ such trajectories. Convex plane billiards were studied by G. Birkhoff, and the case of higher dimensional billiards is considered in our previous papers. We apply a topological approach based on the calculation of cohomology of certain configuration spaces of points on 2-sphere.

1. Introduction

Given a smooth strictly convex closed hypersurface $X^m \subset \mathbf{R}^{m+1}$, consider the billiard system inside the convex body T (the billiard table), bounded by X. One views the billiard ball as a point which moves freely inside T with the elastic reflections in the boundary $X = \partial T$ subject to the familiar law of geometric optics: the incoming and the outgoing trajectories lie in the same 2-plane with the normal at the impact point, and the angle of reflection equals the angle of incidence. Billiard trajectories are extrema of the length functional.

The problem of periodic billiard trajectories is related to the closed geodesics problem on Riemannian manifolds. Unlike the thoroughly studied latter problem (see [8]), the problem of estimating the number of periodic trajectories in convex billiards did not receive much attention. The first results are due to G. Birkhoff [2] who studied periodic billiard trajectories in convex plane billiards (when $m = 1$), known also as "Birkhoff billiards". The next step was made by I. Babenko [1] about 10 years ago who considered convex billiards in 3-space. This case is technically the hardest, and the paper [1] contained an error in cohomological computations (to quote R. Bott [3] on the closed geodesics problem: "For some reason, the large literature on this subject teems with mistakes"). The case of convex billiards in

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m-dimensional spaces with $m \geq 3$ was studied in our recent paper [5]. A relevant result is as follows.

Theorem 1 ([5]). *Let $n \geq 3$ be an odd number and $X^m \subset \mathbf{R}^{m+1}$, $m \geq 3$, be a generic smooth strictly convex closed hypersurface. Then the number of distinct n-periodic billiard trajectories inside X is not less than $(n-1)m$.*

Note that the dihedral group D_n acts on n-periodic billiard trajectories in a natural way, and the above theorem, as well as the results below, estimates the number of distinct D_n-orbits of n-periodic trajectories.

The techniques of [5] do not apply to the case $m = 2$.

A weaker estimate (which however holds for *all* strictly convex billiard tables without any genericity assumptions) is obtained in [7]:

Theorem 2 ([7]). *Let $n \geq 3$ be an odd prime and X be a smooth strictly convex closed surface in 3-space. Then the number of distinct D_n-orbits of n-periodic billiard trajectories inside X is not less than $(n+1)/2$.*

The goal of the present paper is to show that the result of Theorem 1 holds for $m = 2$ as well (it does, for $m = 1$, as follows from Birkhoff's work [2]). Our main result is as follows.

Theorem 3. *Let $n \geq 3$ be an odd number and let X be a generic smooth strictly convex closed surface in 3-space. Then the number of distinct D_n-orbits of n-periodic billiard trajectories inside X is not less than $2(n-1)$.*

The meaning of the genericity assumption is explained in the next section.

2. Cyclic configuration spaces

Let X be a topological space and n be a positive integer. Denote by $G(X, n)$ the subspace of the Cartesian power $X^{\times n} = X \times X \times \cdots \times X$ consisting of all configurations $(x_1, x_2, \ldots, x_n)$ such that $x_i \neq x_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $x_n \neq x_1$. The space $G(X, n)$ is called *the cyclic configuration space*, see [5].

The dihedral group D_n acts naturally on the cyclic configuration space $G(X, n)$. One may think of D_n as the group of symmetries of a regular n-gon. Identifying D_n with the subgroup of permutations of the set of vertices $1, 2, \ldots, n$, one describes the action of D_n on $G(X, n)$ as follows:

$$\tau \cdot (x_1, \ldots, x_n) = (x_{\tau(1)}, \ldots, x_{\tau(n)}) \quad \text{for} \quad \tau \in D_n, \ (x_1, \ldots, x_n) \in G(X, n).$$

The action of D_n is free if and only if n is prime.

The strategy of the proof of Theorem 3 (as well as two other theorems stated in the Introduction) is as follows. As was already mentioned, n-periodic billiard trajectories inside the billiard table T are critical points of the perimeter length function L_X on the space of n-gons, inscribed in $X = \partial T$. The space of inscribed n-gons is the cyclic configuration space $G(X, n)$. The genericity assumption in Theorem 3 means that L_X is a Morse function on $G(X, n)$. This function is invariant under the action of the dihedral group D_n. Although the space $G(X, n)$ is not compact, one can show that the number of critical D_n-orbits of L_X is bounded below by the rank of the D_n-equivariant cohomology space of the cyclic configuration space – see [5], §4.

More precisely, we will use Proposition 4.5 from [5]. Since X is topologically the 2-sphere, our goal is to find the dimension of the graded vector space $H^*_{D_n}(G(S^2, n); \mathbf{Z}_2)$. The answer is given in the next theorem which, therefore, implies Theorem 3.

Theorem 4. *For odd* $n \geq 3$, *the Poincaré polynomial of* $H^*_{D_n}(G(S^2, n); \mathbf{Z}_2)$ *equals*

$$(1 + t^2) \cdot (1 + t + t^2 + \cdots + t^{n-2}),$$

and the sum of the respective Betti numbers is $2(n - 1)$.

This result complements the computation of the rings $H^*_{D_n}(G(S^m, n); \mathbf{Z}_2)$ for $m \geq 3$ and odd n in [5] and the rings $H^*(G(S^m, n); \mathbf{k})$ for various values of m and various fields $\mathbf{k}$ in [7]. However, the results in [7] do not cover the case $m = 2$ and $\mathbf{k} = \mathbf{Z}_2$. The calculations of [5], [6], [7] are based on a spectral sequence computing the cohomology algebra of the cyclic configuration space $G(X, n)$ for an arbitrary manifold X, developed in [5]; this spectral sequence is similar to the Totaro spectral sequence [9] for the usual configuration spaces.

3. Cohomological computations

In this section we prove Theorem 4. We will use some results from our previous papers [5–7]. Recall that n is assumed to be odd.

Let $T : G(S^2, n) \to G(S^2, n)$ be the reflection with respect to the first point

$$T(x_1, x_2, \ldots, x_n) = (x_1, x_n, x_{n-1}, \ldots, x_2), \quad T^2 = \mathrm{Id}.$$

Clearly T has no fixed configurations. Denote by $G' = G(S^2, n)/T$ the quotient manifold.

Let $\mathbf{Z}_2 \subset D_n$ denote the cyclic subgroup of the dihedral group D_n generated by T. We want to prove that $H^*_{D_n}(G(S^2, n); \mathbf{Z}_2)$ is isomorphic to $H^*_{\mathbf{Z}_2}(G(S^2, n); \mathbf{Z}_2) \simeq H^*(G'; \mathbf{Z}_2)$, and we will compute below the latter space. We refer the reader to [3] and [4] for the definition and main properties of equivariant cohomology. Consider the following commutative diagram

$$\begin{array}{ccc} E\mathbf{Z}_2 \times_{\mathbf{Z}_2} G(S^2, n) & \to & ED_n \times_{D_n} G(S^2, n) \\ \downarrow G(S^2, n) & & \downarrow G(S^2, n) \\ B\mathbf{Z}_2 & \to & BD_n \end{array}$$

and the Serre spectral sequences with $\mathbf{Z}_2$ coefficients of the fibrations represented by the vertical arrows (where $EG \to BG$ denotes the universal bundle of a group G).

We obtain a homomorphism of spectral sequences, such that the homomorphism of the E^{∞}-terms is compatible with the homomorphism

$$H^*_{D_n}(G(S^2, n); \mathbf{Z}_2) \to H^*_{\mathbf{Z}_2}(G(S^2, n); \mathbf{Z}_2), \tag{3.1}$$

induced by the upper horizontal map in the diagram above.

Lemma 1. *The homomorphism (3.1) is an isomorphism.*

Proof. On the level of E_2-terms, one has a homomorphism

$$\begin{aligned} E_2^{p,q} &= H^p(D_n; H^q(G(S^2, n); \mathbf{Z}_2)) \to E'^{p,q}_2 \\ &= H^p(\mathbf{Z}_2; H^q(G(S^2, n); \mathbf{Z}_2)). \end{aligned} \tag{3.2}$$

We claim that (3.2) is an isomorphism. The claim would follow from Proposition 10.4 of Chapter 3, [4] if one shows that the dihedral group D_n acts trivially on the vector space

$$H^p(\mathbf{Z}_2; H^q(G(S^2, n); \mathbf{Z}_2)).$$

According to Theorem 4 and Remark 3.3 from [5], the cyclic group $Z_n \subset D_n$ acts trivially on $H^q(G(S^2, n); \mathbf{Z}_2)$ and the dihedral group D_n acts trivially on $H^q(G(S^2, n); \mathbf{Z}_2)$ for q equal $0, 1, n-1, n$. Moreover, for $1 < q < n-1$ the cohomology space $H^q(G(S^2, n); \mathbf{Z}_2)$ is two-dimensional and D_n acts trivially on a one-dimensional subspace and on its quotient space. It follows, that for any value of q between 1 and $n-1$ there exist two possibilities: either D_n acts trivially on $M = H^q(G(S^2, n); \mathbf{Z}_2)$, or M, viewed as a $\mathbf{Z}_2[\mathbf{Z}_2]$-module, is the unique nontrivial extension $0 \to \mathbf{Z}_2 \to M \to \mathbf{Z}_2 \to 0$. One checks directly, using the definition of group cohomology, that in the second case $H^p(\mathbf{Z}_2; M) = 0$ for all $p > 0$. In the second case $H^0(\mathbf{Z}_2; M)$ is isomorphic to the subspace of $\mathbf{Z}_2$-invariants in M, which clearly coincides with the subspace of D_n-invariants in M. Hence (3.2) is an isomorphism for all p and q.

The comparison theorem for spectral sequences implies an isomorphism

$$H^*_{D_n}(G(S^2, n); \mathbf{Z}_2) \to H^*_{\mathbf{Z}_2}(G(S^2, n); \mathbf{Z}_2),$$

as claimed. □

Now we will study the space $H^*(G'; \mathbf{Z}_2)$. Fix a point $A \in S^2$. Along with the cyclic configuration space $G(S^2, n)$, consider its subspace G_A consisting of the configurations $(x_1, x_2, \dots, x_n) \in G(S^2, n)$ with $x_1 = A$. Clearly, G_A is T-invariant; set $G'_A = G_A/T$.

The standard action of $SO(3)$ on S^2 by rotations induces an action of $SO(3)$ on the cyclic configuration space $G(S^2, n)$

$$SO(3) \times G(S^2, n) \to G(S^2, n).$$

The restriction of this map to G_A gives a continuous map

$$p : SO(3) \times G_A \to G(S^2, n). \tag{3.3}$$

and we claim that it is a fibration. To prove this we will show that (3.3) is induced from the standard fibration

$$q : SO(3) \to S^2, \quad \text{where} \quad q(R) = R(A), \quad R \in SO(3), \tag{3.4}$$

by the map $f : G(S^2, n) \to S^2$ given by $f(x_1, \dots, x_n) = x_1$. The total space E of the induced fibration is the space of all pairs $(R, (x_1, \dots, x_n))$ with $R(A) = x_1$, where $R \in SO(3)$ and $(x_1, \dots, x_n) \in G(S^2, n)$. The map $(R, (x_1, \dots, x_n)) \mapsto (R, (A, R^{-1}(x_2), \dots, R^{-1}(x_n))$ is a homeomorphism $E \to SO(3) \times G_A$. This proves that (3.3) is the induced fibration.

The fiber S^1 of the fibration (3.3) can be described as follows. Let $c = (A, x_2, \dots, x_n) \in G_A$ be a fixed configuration. If a rotation $R \in SO(3)$ preserves c then R is identical (if n is even the configuration $(A, -A, A, -A, \dots)$ is invariant under rotations about the axis $(A, -A)$). Then the fiber $p^{-1}(c)$ is identified with the set of pairs $(R_\phi, R_{-\phi}c)$, where $R_\phi \in SO(3)$ denotes the rotation through angle ϕ about A. Factorizing by T, one obtains the fibration

$$p' : SO(3) \times G'_A \to G' \tag{3.5}$$

with fiber S^1. As above, fibration (3.5) is induced from (3.4) by the map $G' \to S^2$ sending orbit of any configuration $(x_1, x_2, \dots, x_n) \in G(S^2, n)$ to $x_1 \in S^2$.

Consider the Serre spectral sequence of (3.5) with $\mathbf{Z}_2$ coefficients. The cohomology of the total space

$$H^*(SO(3) \times G'_A; \mathbf{Z}_2) \simeq H^*(SO(3); \mathbf{Z}_2) \otimes H^*(G'_A; \mathbf{Z}_2)$$

is as follows.

It is well known that $H^*(SO(3); \mathbf{Z}_2) = \mathbf{Z}_2[v]/(v^4)$ is the truncated polynomial algebra with a single 1-dimensional generator v satisfying the relation $v^4 = 0$. The cohomology algebra $H^*(G'_A; \mathbf{Z})$ with integral coefficients was described in Theorem 12 of [7]. It has generators

$$\delta_i \in H^{4i}(G'_A; \mathbf{Z}) \simeq \mathbf{Z}, \quad \text{where} \quad i = 1, 2, \dots$$

and two other generators

$$a \in H^1(G'_A; \mathbf{Z}) \simeq \mathbf{Z}, \quad \text{and} \quad b \in H^3(G'_A; \mathbf{Z}) \simeq \mathbf{Z}_2,$$

which satisfy the following defining relations:

$$\begin{array}{ll} \delta_i \delta_j = \binom{2i+2j}{2i} \cdot \delta_{i+j}, & \delta_{[(n+1)/4]} = 0, \\ b^2 = 0 & 2b = 0, \\ a^2 = 0, & ab = 0, \\ \delta_k b = 0 \quad (\text{only for } n = 4k+3). & \end{array}$$

An additive basis of $H^*(G'_A; \mathbf{Z}_2)$ is given by mod 2 reductions of the classes δ_i of degree $4i$ and classes $\delta_i a$ of degree $4i + 1$; moreover, according to the universal coefficient formula, each integral class $\delta_i b$ of order 2 produces two classes in $\mathbf{Z}_2$-cohomology, one of degree $4i + 2$ and one of degree $4i + 3$.

We claim that the Poincarè polynomial of $H^*(G'_A; \mathbf{Z}_2)$ equals

$$1 + t + t^2 + \cdots + t^{n-2}. \tag{3.6}$$

To prove this, consider two cases: $n = 4k + 3$ and $n = 4k + 1$. In the first case the mod 2 reductions of the generators δ_i and $\delta_i a$ produce the Poincaré polynomial

$$(1 + t)(1 + t^4 + t^8 + \cdots + t^{4k}),$$

and the classes $\delta_i b$ contribute

$$(t^2 + t^3)(1 + t^4 + t^8 + \cdots + t^{4k-4}) \tag{3.7}$$

to the Poincaré polynomial. Summing these two polynomials gives (3.6). In the second case, when $n = 4k + 1$, the contribution of the generators δ_i and $\delta_i a$ is

$$(1 + t)(1 + t^4 + \cdots + t^{4k-4});$$

the contributions of the classes $\delta_i b$ is (3.7), and the total is again (3.6).

Next, consider the cohomological spectral sequence of the fibration (3.5). The term E_2 consists of two rows, and the only possibly non-trivial differential is d_2. We claim that this differential vanishes. Assuming this claim, the Poincarè polynomial of the base equals the Poincarè polynomial of the total space divided by the Poincarè polynomial of the fiber. We obtain the Poincarè polynomial of $H^*(G'; \mathbf{Z}_2)$:

$$\frac{(1 + t + t^2 + t^3) \cdot (1 + t + t^2 + \cdots + t^{n-2})}{1 + t} = (1 + t^2) \cdot (1 + t + t^2 + \cdots + t^{n-2})$$

as stated in Theorem 4.

It remains to prove that $d_2 = 0$. Let $s \in E_2^{0,1} = H^1(S^1; \mathbf{Z}_2)$ be the generator. It suffices to show that $d_2(s) = 0$. The class s is transgressive, and $d_2(s)$ is the mod 2 reduction of the Euler class of fibration (3.5). Fibration (3.5) is induced from fibration (3.4), cf. above. The latter is the unit tangent bundle of the 2-sphere. The Euler number of S^2 equals 2, and the Euler class of (3.4) vanishes in $H^2(S^2; \mathbf{Z}_2)$. By functoriality of characteristic classes, the mod 2 reduction of the Euler class of (3.5) is trivial.

This completes the proof of Theorem 4.

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