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COMPUTATION OF THE BENNEQUIN INVARIANT OF A LEGENDRE CURVE FROM THE GEOMETRY OF ITS FRONT

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1. Let M^3 be a contact manifold and γ a contractible Legendre curve in it. The Bennequin invariant $\ell(\gamma)$ is defined as the linking coefficient of γ with the curve obtained by a small shift of γ along a transversal to it in contact planes [1]. We shall compute the Bennequin invariant in the following contact manifolds: $M_1 \simeq S^1 \times \mathbf{R}^2$, the manifold of unoriented contact elements; $M_n \simeq S^1 \times \mathbf{R}^2$, its n-sheeted covering; $M_\infty \simeq \mathbf{R}^3$, its universal covering. If x, y are coordinates in $\mathbf{R}^2$ and α is the angular coordinate, then the contact form is $\lambda = \sin\alpha dx - \cos\alpha dy$.

The image of γ under projection of the Legendre bundle $M_n \to \mathbf{R}^2$ is called its front Γ. In general position Γ is a curve with double points and cusps (beaks), and to each such curve there corresponds a unique Legendre curve in M_1. The degree of the front is defined as the degree of its Gaussian image. If Γ is a curve of degree zero, then γ is contractible in M_1 and it can be lifted to each of the M_n.

2. Let Γ be a front of degree zero. Let us fix its orientation and clothing, that is, a nonzero normal vector field. On Γ there is uniquely defined the function of the angle between the clothing vector and a fixed direction. Suppose that the tangent vectors to Γ at a double point A form a positive frame, and that angles α and β correspond to the normal vectors. We ascribe to A the sign +1 if $\alpha < \beta$, and −1 if $\beta < \alpha$. In the computation in M_n we ascribe to A the multiplicity $2[|\alpha-\beta|/\pi n]+1$. For example, to the double point of the "figure eight" there corresponds −1 in M_∞. We denote by h the sum of the multiplicities of the double points of the front Γ and by c the (even) number of beaks.

3. In the computation in M_n we shall take as a double tangent only a double tangent for which the angles of the normal vectors at the points of contact are equal $\bmod\ \pi n$. For example, the "figure eight" in M_∞ does not have double tangents. We say that a point of the front is even if the tangent and normal at the point form a positive frame, and odd otherwise. In passing through a beak the parity changes. We ascribe a sign to a double tangent as follows: if the points of contact are of the same parity and the germs of the curve lie on the same side of the tangent, we have +1, and if they are on opposite sides, then −1; if the points of contact are of different parity, then the signs are opposite. We denote by t the sum of the multiplicities of the double tangents and by i the (even) number of points of inflexion of the front Γ.

4. In the computation in M_n we shall define a diameter as a chord perpendicular to Γ for which the angles of the normal vectors to Γ at its ends are equal $\bmod\ \pi n$. For example, an ellipse does not have diameters in M_2, and the "figure eight" has one diameter in M_∞. We ascribe a sign to a diameter as follows: if the ends are of the same parity and the germs of Γ are transformed into each other by convexity, the sign is −1; under a deformation of the germs the sign changes for each coincidence of their centers of curvature. If the ends of a diameter are of different parity, then the sign is opposite. Thus, the sign corresponds to the stability of the diameter as a billiards trajectory in Γ. We note that to the diameters of Γ there correspond characteristic chords of γ, that is, segments of the trajectories of the field of kernels of the 2-form $d\lambda$ of Sec. 1 with ends on the curve γ. We denote the sum of the multiplicities of the diameters by d.

5. In general position Γ does not have vertical double tangents or beaks with vertical tangents. In the computation in M_n we ascribe a sign to vertical chords with equal $(\bmod\ \pi n)$ angles of the normal vectors to Γ at the ends as follows: if the ends of the chord have the

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same parity and the germs of Γ are transformed into each other by convexity, then we have -1; under a deformation of the germs the sign changes with each coincidence of the values of the curvatures. If the ends of the chord have different parities, then the sign is opposite. We denote by k the sum of the multiplicities of the vertical chords of Γ. In other words, k is the number of self-intersections (with signs) of the projection of γ on the plane (x, α).

We ascribe to beaks formed with the cusp-point to the right the sign -1, and those with the cusp-point to the left the sign $+1$. We denote by ℓ the sum of the multiplicities of the beaks and by m the number of vertical tangents of Γ lying locally to the right of the front.

THEOREM. $\ell(\gamma) = h - c/2 = t - i/2 = d = k - m + \ell/2$.

For example, to the "figure eight" in M_∞ there corresponds $\ell(\gamma) = -1$.

6. If Γ is a plane clothed front of nonzero degree r, then the corresponding Legendre curve γ in M_2 is uncontractible. We take for Γ' an r-fold small circle at infinity; then γ is homologous to γ' in M_2 and we can look for the index of intersection of the membrane spanned by γ and γ' with the curve obtained from γ by a small shift along a transversal in the contact planes. We call this index the Bennequin affine invariant $A\ell(\gamma)$.

THEOREM. $A\ell(\gamma) = t - i/2 - r^2 = d - r^2 = k - m + \ell/2$.

7. In the situation of the previous section we can look for the index of intersection of the membrane spanned by γ and γ' with the Legendre curve corresponding to the front Γ_1. Suppose that Γ and Γ_1 are nonintersecting immersed curves, and that Γ lies inside one of the components of the complement to Γ_1. Let $\deg \Gamma = p$, $\deg \Gamma_1 = q$, and let s be the degree of Γ_1 with respect to any point of Γ.

Proposition. The number of common tangents of Γ and Γ_1 and the number of their common perpendiculars $\geqslant 4\,|p(q-s)|$.

Here the tangents and perpendiculars are understood as for computation in M_1, that is, $\bmod \pi$.

8. Let Γ be an affine front on the sphere S^2, that is, a curve lying in a hemisphere. The degree of Γ is defined mod 2 and the corresponding Legendre curve γ in the space of oriented contact elements $\mathbf{R}P^3$ is contractible if $\deg \Gamma$ is even. In this case γ can also be lifted to S^3. Suppose that Γ is represented in the affine chart by a front of degree 2r. In the computation in $\mathbf{R}P^3$ the angles of the normals to Γ at the points of contact of the double tangents coincide $\bmod 2\pi$, and in the computation in S^3 they coincide $\bmod 4\pi$. Similarly with diameters and chords.

THEOREM. $\ell(\gamma) = t - i/2 - r^2 = d - r^2 = k - m + \ell/2 + r^2$.

For example, to a cardioid in S^2 there corresponds $\ell(\gamma) = -1$.

9. Remarks. (i) The identities of Theorem 5 are connected with an assertion [2, 3] that can also be proved by the method of Sec. 6: the number of double points of an immersed plane curve is equal to the sum of the multiplicities of the double tangent minus half the number of points of inflexion.

(ii) We consider the self-linking coefficient of a curve, which arises from the metric: the curve is shifted along the field of its principal normals [4-6]. In the case of the standard S^3 the Bennequin invariant coincides with this coefficient [1], and in $\mathbf{R}^3$ with coordinates x, y, α the Bennequin invariant differs from the metric self-linking coefficient by half the number of points of inflexion of the front.

10. If $L^n \subset M^{2n+1}$ is a Legendre submanifold of a contact manifold, homologous to zero, then the definition of $\ell(L)$ as the linking of L with the manifold obtained from L by a small shift along the field of characteristics retains it sense (although there is interest in only odd n, since if n is even we have $\ell(L) = \chi(L)/2$, see [6]). Let $X^n \subset \mathbf{R}^{n+1}$ be an embedded hypersurface such that the degree of its Gaussian image is equal to zero. The corresponding Legendre submanifold L of the space of unoriented or oriented contact elements $\mathbf{R}P^n \times \mathbf{R}^{n+1}$ or $S^n \times \mathbf{R}^{n+1}$ is homologous to zero.

THEOREM. In both cases $\ell(L) = 0$.

As in Sec. 4, $\ell(L)$ is computed from the diameters of X. In the computation in $S^n \times \mathbf{R}^{n+1}$ the oriented normals of X coincide at the ends of a diameter. If A and B are the second quadratic forms of X at the ends of a diameter and d is its length, then the diameter is taken with the sign sgn det(A − B − dAB).

COROLLARY. The number of diameters (all of them or those with coincident orientations of X at the ends) is even, and if signs are taken into account it is zero.

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COHOMOLOGY OF THE COMMUTATOR SUBGROUP OF THE BRAID GROUP

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In this note the complex cohomology of the commutator subgroup of E. Artin's braid group is computed. The classifying space of this commutator subgroup is realized in the space of (complex) polynomials of a given degree as a nonsingular discriminant level surface. Our result can be applied in the study of characteristic classes of algebraic functions (see [1]) and in description of vector fields tangent to the discriminant zero level (see [2]).

1. Statement of the Result. Recall that the braid group B(n) of n threads is generated by n − 1 generators $\sigma_1,\ldots,\sigma_{n-1}$ subject to the relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|j - i| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for i = 1,...n − 2. The classifying space for the group B(n) is given by the space of unordered families of n distinct complex numbers or the space $V^{n-1} - \Delta^{-1}(0)$, where V^{n-1} is the space of (complex) polynomials of the form $z^n + v_1z^{n-2} + \ldots + v_{n-1}$, and $\Delta: V^{n-1} \to \mathbf{C}$ is the discriminant (cf. [1]).

The integral cohomology of the braid group is known (cf. [1, 3-5]). In particular, $H^1(B(n); \mathbf{Z}) = \mathbf{Z}$ and the groups $H^q(B(n), \mathbf{Z})$ are finite for q > 1.

The commutator subgroup B'(n) of the group B(n) coincides with the kernel of the epimorphism $\varepsilon: B(n) \to \mathbf{Z}$, mapping σ_i into 1. The classifying space of the group B'(n) may be given by a nonsingular level curve of the function Δ or a natural Z-covering over $V^{n-1} - \Delta^{-1}(0)$ (a Riemannian surface of the function log Δ). In this note we compute the dimensions of the spaces $H^q(B'(n); \mathbf{C})$, implicitly using the latter model of the classifying space.

THEOREM. If q = n(k − 2)/k, where k is a divisor of the number n distinct from 1, or q = (n − 1)(k − 2), where k is a divisor of the number n − 1 distinct from 1, then $\dim H^q(B'(n); \mathbf{C}) = \varphi(k)$, where φ is the Euler function [φ(k) is the number of natural numbers less than k which are relatively prime with k]. In the remaining cases $H^q(B'(n); \mathbf{C}) = 0$. In particular,

$$H^q(B'(n);\mathbf{C}) = 0, \text{ if } 0 < q < \left[\frac{n}{3}\right] \text{ or } q > n-2,$$

$$\dim H^{n-3}(B'(n);\mathbf{C}) = \varphi(n-1), \quad \dim H^{n-2}(B'(n);\mathbf{C}) = \varphi(n).$$

The proof of this theorem is contained in Secs. 2 through 5; it includes an explicit description of the generators of the cohomologies in question.

Note that the (integral) one-dimensional cohomologies of the group B'(n) have been computed by Gorin and Lin [6] in 1969. They proved that $B'(n)/B''(n) = \mathbf{Z} \oplus \mathbf{Z}$ for n = 3, 4 and B"(n) = B'(n) for n > 4. The classifying space of the group B'(3) has a very simple structure: it is a punctured torus. The classifying space of the group B"(4) was studied in [2];

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