

1. Definition and Global Properties of Lagrangian Foliations

We call a manifold M^{2n+r} quasisymplectic if there is given a closed 2-form ω of constant rank $2n$ on it, and quasicontract if there exists a nondegenerate 1-form λ , such that $\omega = d\lambda$ and the distribution $\text{Ker } \lambda$ is transverse to $\text{Ker } \omega$. A foliation \mathcal{F} on a quasisymplectic manifold M^{2n+r} is called Lagrangian if $\omega|_{\mathcal{F}} = 0$ and $\dim \mathcal{F} = n + r$ (i.e., \mathcal{F} is maximal isotropic).

Of course a symplectic manifold is a special case of quasisymplectic manifold. An example of a quasicontract manifold with a Lagrangian foliation is a contact manifold (M^{2n+1}, λ) and a foliation \mathcal{F}^{n+1} on it such that the characteristic field $\text{Ker } d\lambda$ is tangent to \mathcal{F} and \mathcal{F} cuts out a Lagrangian subbundle in the symplectic distribution $\text{Ker } \lambda$. In particular, the horocyclic foliation on the manifold of contact elements of a flat torus or of a surface of genus ≥ 2 with metric of constant negative curvature (a leaf is formed by parallel contact elements) is such.

The non-simple connectivity of the contact manifold carrying the Lagrangian foliation in the preceding example is not accidental.

Proposition. There do not exist Lagrangian foliations on a closed three-dimensional contact manifold with finite fundamental group.

This proposition follows from the following assertion since the characteristic field has zero divergence.

THEOREM. A vector field with divergence zero on a closed three-dimensional manifold with finite fundamental group is not contained in any two-dimensional foliation.

The discrete analog of this theorem is also true: the field is replaced by a diffeomorphism preserving volume and carrying each leaf of the foliation into itself. The proof (proposed by V. L. Ginzburg) consists of applying Novikov's theorem on the existence of a Reeb component of the foliation and applying Poincare's recurrence theorem to it.

2. Local Structure of Lagrangian Foliations

The distribution $\text{Ker } \omega$ is integrable and ω defines a symplectic structure on the manifold $M/\text{Ker } \omega$ with respect to which the foliation $\mathcal{F}/\text{Ker } \omega$ is Lagrangian. The pair $(M/\text{Ker } \omega, \mathcal{F}/\text{Ker } \omega)$ is locally symplectomorphic to T^*U^n fibred by p-spaces. Thus, a Lagrangian foliation on a quasisymplectic manifold is locally isomorphic to the product of the cotangent bundle by an r-dimensional space. Analogously, in the case of a quasicontract manifold the distribution $\mathcal{F} \cap \text{Ker } \lambda$ is integrable and the Lagrangian foliation \mathcal{F} is locally isomorphic to the product of the manifold of 1-jets J^1U^n with coordinates (p, q, u) , fibred by $(n + 1)$ -dimensional fibres $q = \text{const}$, and an $(r - 1)$ -dimensional space.

We note that ω defines an isomorphism of the normal bundle $\nu(\mathcal{F}) \simeq (\mathcal{F}/\text{Ker } \omega)^*$. For Lagrangian foliations one has the analog of Bott's theorem:

Proposition. Monomials of degree $> 2 \text{codim } \mathcal{F}$ in the Pontryagin classes of the bundle $\nu(\mathcal{F})$ and the class $[\omega] \in H^2(M)$ are equal to zero.

3. Construction of Characteristic Classes

As is known, a framed foliation \mathcal{F} on M of codimension n defines a W_n -structure, i.e., a 1-form with values in the Lie algebra of formal vector fields on R^n . This lets one construct the characteristic homomorphism $H^*(W_n) \rightarrow H^*(M)$. For Lagrangian foliations on quasi-

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symplectic manifolds the role of the algebra W_n is played by the subalgebra \mathfrak{g}_n of the algebra of Hamiltonian fields on T^*U^n preserving the cotangent bundle, and in the quasiconformal case by the algebra $\tilde{\mathfrak{g}}_n$ of fields in J^1U^n preserving the contact form $p dq - du$ and the fibration $q = \text{const}$. We denote by F_n the space of formal power series in \mathbb{R}^n , on which W_n acts by Lie differentiation, and by \mathfrak{m}_n its quotient module by the constants.

Proposition. \mathfrak{g}_n is a central extension of W_n by \mathfrak{m}_n , and $\tilde{\mathfrak{g}}_n$ is a central extension of W_n by F_n :

$$0 \rightarrow \mathfrak{m}_n \rightarrow \mathfrak{g}_n \rightarrow W_n \rightarrow 0; \quad 0 \rightarrow F_n \rightarrow \tilde{\mathfrak{g}}_n \rightarrow W_n \rightarrow 0.$$

The construction of a \mathfrak{g}_n -structure on a framed Lagrangian foliation repeats the familiar construction for W_n with the following changes: one considers the manifolds S^1 of i -jets of submersions of M and T^*U^n mapping the leaves of \mathcal{F} into leaves of the cotangent bundle and defining symplectomorphisms $M/\text{Ker} \simeq T^*U^n$. The fibration $S^1 \rightarrow S^0$ is homotopy equivalent to the bundle of frames $\nu(\mathcal{F})$ and $TS^\infty \simeq \mathfrak{g}_n$, which lets one construct a \mathfrak{g}_n -valued 1-form on M .

Here is a more geometric description of this \mathfrak{g}_n -structure. We fix at each point $x \in M$ a Lagrangian transversal T_x to \mathcal{F} and we identify all T_x with \mathbb{R}^n . For a point y close to x the area element of T_y is defined in T^*U as the graph of the differential of the generating function g_y , and the leaves of \mathcal{F} are defined by the germ of the diffeomorphism $\varphi_y: \mathbb{R}^n = T_x \rightarrow T_y = \mathbb{R}^n$. The differential of the map $y \mapsto (g_y, \varphi_y)$ defines a 1-form on M with values in $\mathfrak{m}_n + \tilde{W}_n = \mathfrak{g}_n$. Analogously in the quasiconformal case the Lagrangian foliation defines a $\tilde{\mathfrak{g}}_n$ -structure (the transversal to the foliation is defined as the 1-graph of a function in J^1U^n).

Thus, the characteristic classes of framed Lagrangian foliations correspond to elements of the groups $H^*(\mathfrak{g}_n)$ and $H^*(\tilde{\mathfrak{g}}_n)$. These characteristic classes are invariant with respect to concordance of Lagrangian foliations, and the corresponding characteristic numbers are bordism invariants.

4. Calculation of the Characteristic Classes

We denote by Y_n the preimage of the $2n$ -skeleton of the base in the product of the universal $U(n)$ -bundle by the trivial bundle $BU(1) \rightarrow BU(1)$; by Z_n the preimage of the $2n$ -skeleton of the base in the universal $U(n) \times U(1)$ -bundle.

THEOREM. $H^*(\mathfrak{g}_n) \simeq H^*(Y_n)$; $H^*(\tilde{\mathfrak{g}}_n) \simeq H^*(Z_n)$.

The proof, as in the well-known case of W_n , is extracted from the Serre-Hochschild spectral sequence of the pair $\mathfrak{gl}_n \subset \mathfrak{g}_n$.

The cohomology of the spaces Y_n and Z_n is calculated in the standard way. It follows in particular from this calculation that $H^{\leq 2n}(\mathfrak{g}_n)$ is generated by powers of a two-dimensional class which corresponds in $H^*(M)$ to the class of the form ω , and that $H^{\leq 2n}(\tilde{\mathfrak{g}}_n) = 0$.

For $n = 1$ we give a direct construction in the spirit of Godbillon-Vey. Let a Lagrangian foliation be defined by a 1-form α with $d\alpha = \alpha \wedge \eta$ (integrability condition). $H^*(\mathfrak{g}_1)$ has one two-dimensional and two three-dimensional generators and the corresponding characteristic classes are defined by the forms ω , $\omega \wedge \eta$, and $\eta \wedge d\eta$; the last is the Godbillon-Vey class. In the quasiconformal case $\dim H^*(\tilde{\mathfrak{g}}_1) = 5$ and the corresponding classes are defined by the following forms: $\lambda \wedge d\lambda$, $\eta \wedge d\lambda$, $\eta \wedge d\eta$, $\lambda \wedge \eta \wedge d\lambda$, $\lambda \wedge \eta \wedge d\eta$. It is interesting to verify directly the independence of these classes from the choice of the forms α and η .

We give an example of a Lagrangian foliation with nontrivial characteristic classes. We denote by G the three-dimensional group generated by hyperbolic rotations and parallel translations of the plane. A basis for the invariant forms consists of 1-forms $\alpha_1, \alpha_2, \alpha_3$ with $d\alpha_1 = \alpha_1 \wedge \alpha_3$, $d\alpha_2 = \alpha_3 \wedge \alpha_2$, $d\alpha_3 = 0$. These forms are also defined on the compact quotient by a discrete subgroup G/π ; α_1 defines a foliation which is Lagrangian with respect to the 2-form $\omega = \alpha_1 \wedge \alpha_2$. Here $[\omega] \neq 0$, $[\eta \wedge d\eta] = 0$, and $[\omega \wedge \eta] = [\alpha_1 \wedge \alpha_2 \wedge \alpha_3] \neq 0$.

5. Cf. [1] for all references to the theory of foliations. The author thanks A. T. Fomenko for posing the problem and A. B. Givental', V. L. Ginzburg, and D. B. Fuks for their interest and help.

LITERATURE CITED

1. D. B. Fuks, Cohomology of Infinite-Dimensional Lie Algebras [in Russian], Nauka, Moscow (1984).