TWO REMARKS ON THE ASYMPTOTIC HOPF INVARIANT

S. L. Tabachnikov

UDC 517.9

1. Asymptotic Hopf Invariant. A vector field u on a three-dimensional manifold with a volume from μ is said to be homologous to zero if $i_u\mu = d\alpha$ for a certain 1-form α. The number $\int \alpha \wedge d\alpha$ (independent of the choice of α) is called the asymptotic Hopf invariant of the field u. The asymptotic linking coefficient L(u, v) of the vector fields that are homologous to zero is defined in the same manner; it is the number $\int \alpha \wedge d\beta$, where the forms α and β correspond to the two given fields. Arnol'd [1] has shown that the asymptotic Hopf invariant can be computed geometrically: We select points x and y; draw through them segments of trajectories of the field for time T; close "in short" each of the curves; compute their linking coefficient; take average with respect to the pairs (x; y); divide by T^2; and take limit on $T \to \infty$. The linking coefficient of two fields is analogously computed.

In this note we set forth two variations on this theme and give a series of unsolved problems.

2. Godbillon–Vey Class. Let $\mathcal{F}$ be a cooriented foliation of codimension 1. We define its 1-form α; then $d\alpha = \alpha \wedge \eta$ for a certain 1-form η. The form $\eta \wedge d\eta$ is closed and its homology class (independent of the choice of α and η) is called the Godbillon–Vey class of the foliation. In the case where the total manifold of the foliation is three-dimensional, following Sullivan [2] and a verbal remark of V. I. Arnol'd, we interpret this class as the asymptotic Hopf invariant.

We fix a vector field v such that $\alpha(v) = 1$ and denote its flow by φ_t. We choose a volume form μ and define a vector field u by the equation $i_u\mu = \frac{d}{dt}(\varphi_t\alpha) \wedge \alpha|_{t=0}$ This field is tangential to the foliation and gives the axis and the angular rate of rotation of the tangent spaces of the foliation, realized by the flow φ_t. Let us determine the vector field w, giving the rate of change of the field u along φ_t by the equation $i_u\mu = \frac{d^2}{dt^2}(\varphi_t\alpha) \wedge \alpha|_{t=0}$.

Proposition 1. The asymptotic Hopf invariant of the field w is equal to the Godbillon–Vey class.

It is interesting to know whether some of the leading characteristic classes of foliations can be interpreted in this spirit.

Remark. It is interesting to compare the construction of Proposition 1 not entirely with Thruston's natural remark [3] that the Godbillon–Vey class is the "helical wobble" of the foliation. This remark of Thurston is accompanied by a diagram, showing several leaves of the foliation that are spread on the transversal axis, making the leaves a constant angle and around which the leaves are uniformly rotated. This rotation can be given by a turn of the field u, obtained under intersection of a leaf with the leaf, close to it, for a displacement of the latter one along the transversal axis. We have denoted this field by w. It is possible that the asymptotic Hopf invariant of the field w is the "helical wobble"?

3. Asymptotic Bennequin Invariant. Bennequin [4] has defined the invariant $\ell(\gamma)$ of a contractible curve γ, transversal to the contact structure in a three-dimensional manifold, as the linking coefficient of γ with the curve, obtained from γ by a small displacement along a nondegenerate (on the whole manifold) vector field that is tangential to the contact structure. Let us define analogous notion, in which a vector field plays the role of the curve γ.

Let ξ and u be fields on a three-dimensional manifold that are homologous to zero and φ_t be the flow of u. The quantity $\frac{d^2}{dt^2}(L(\varphi_t(\xi), \xi) - L(\xi, \xi))|_{t=0}$ (the first derivative with respect to

---

M. V. Lomonosov Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 24, No. 1, pp. 84-85, January-March, 1990. Original article submitted August 2, 1988.

t is equal to zero) is called the asymptotic Bennequin invariant $\ell(\xi, u)$. Differentiating the obvious equality $L(\varphi_t\xi, \varphi_t\xi) - L(\xi, \xi) = 0$ with respect to t and setting $t = 0$, we get $L(\xi, [\xi, u]) = 0$. Hence the following proposition follows.

Proposition 2.

Let us consider a manifold with the contact form λ and the volume form $\lambda \wedge d\lambda$. Let ξ be the characteristic field [i.e., $i_\xi d\lambda = 0$, $\lambda(\xi) = 1$] and ν be a one-dimensional subbundle of the contact structure. Bennequin [5] has defined the following quadratic form on ν: If u is a section of ν, then $B(u, u) = d\lambda([\xi, u], u)$ (the value at a point x depends only on the value of u(x); we analogously define a quadratic form on the Lagrangian subbundle of the contact structure in arbitrary dimension).

Proposition 3. The linking coefficient of the fields ξ and u is equal to zero and their asymptotic Bennequin invariant is equal to the mean value of the Bennequin form over the whole manifold: $\int B(u, u)\, \lambda \wedge d\lambda$.

Here are two properties of positivity of the invariant ℓ. In both cases the volume form is equal to $\lambda \wedge d\lambda$.

Proposition 4. Let u and v be fields that are tangential to the contact structure and are linearly independent on a set of positive measure. Then $\ell(u, v) > 0$.

Let us consider a standard contact structure in S^3, i.e., an SU(2)-invariant structure. Let λ be a certain 1-form, giving this structure, ξ be the characteristic field, and u be the SU(2)-invriant nondegenerate field, tangential to the contact structure.

Proposition 5. $\ell(\xi, u) < 0$.

It would be interesting to use the asymptotic Bennequin invariant to distinguish nondiffeomorphic contact structures in the same way as their Bennequin invariant distinguishes [4].

4. Two Unsolved Problems. (i) Let ξ be a field on the manifold M^{2n+1} with formal volume μ such that $i_\xi\mu = (d\lambda)^n$ for a certain 1-form λ. Then the asymptotic Hopf invariant $\int \lambda \wedge d\lambda^n$ is defined. Can we not give a geometric method of its computation in the spirit of Sec. 1?

(ii) Let ξ be the characteristic field of a contact 1-form λ on M^{2n+1}, φ_t be its flow, and ν be the Lagrangian subbundle of the contact structure. We compute how many times $\varphi_t(\nu)$ is not transversal to ν on the time segment [0; T]; we average it over the manifold; divide it by T; and take limit as $T \to \infty$. It is appropriate to call the obtained quantity the asymptotic Maslov index. Cannot it be given "de Rahm" definition in terms of differential forms?

The author is deeply thankful to V. A. Arnol'd, D. B. Fuks, and B. A. Khesin for useful discussions.

LITERATURE CITED

1. V. I. Arnold, Selecta, 5, No. 4, 327-347 (1986).
2. D. Sullivan, Invent. Math., 36, 225-255 (1976).
3. W. Thruston, Bull. Am. Math. Soc., 78, No. 4, 511-514 (1972).
4. D. Bennequin, Asterisque, 107-108, 87-161 (1983).
5. D. Bennequin, in: P. Dazord and N. Desolneux-Moulis (eds.), Géométrie Symplectique et de contact: Autour du "Théorème de Poincaré–Birkhoff, Séminare Sud-Rhodanien de Géométrie. III, Hermann, Paris (1984), pp. 1-50.