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Self-dual polygons and self-dual curves

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Abstract We study projectively self-dual polygons and curves in the projective plane. Our results provide a partial answer to problem No 1994-17 in the book of Arnold's problems (2004).

Keywords Projective plane · Projective duality · Polygons · Legendrian curves · Radon curves

Mathematics Subject Classification (2000) 51A99 · 53A20

1 Introduction

The projective plane P is the projectivization of 3-dimensional space V (we consider the cases of two ground fields, $\mathbb{R}$ and $\mathbb{C}$); the points and the lines of P are 1- and 2-dimensional subspaces of V. The dual projective plane P^* is the projectivization of the dual space V^*. Assign to a subspace in V its annihilator in V^*. This gives a correspondence between the points in P and the lines in P^*, and between the lines in P and the points in P^*, called the projective duality. Projective duality preserves the incidence relation: if a point A belongs to a line B in P then the dual point B^* belongs to the dual line A^* in P^*. Projective duality is an involution: $(A^*)^* = A$.

Projective duality extends to polygons in P. An n-gon is a cyclically ordered collection of n points and n lines satisfying the incidences: two consecutive vertices lie on the respective side, and two consecutive sides pass through the respective vertex. We assume that our

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polygons are nondegenerate: no three consecutive vertices are collinear. Thus, to every polygon $L \subset P$ there corresponds the dual polygon $L^* \subset P^*$. A polygon L is called self-dual if there exists a projective map $P \to P^*$ that takes L to L^*.

Projective duality also extends to locally convex smooth curves. A smooth curve $\gamma \subset P$ determines a one-parameter family of its tangent lines, and projective duality takes it to a one-parameter family of points in P^*, the dual curve $\gamma^* \subset P^*$. If γ is locally convex then γ^* is smooth as well. One has $(\gamma^*)^* = \gamma$. Projective duality further extends to a broader class of curves with inflections and cusps, called *wave fronts* (see end of Sect. 5 for a precise definition). Projective duality interchanges inflections and cusps. One defines self-dual curves similarly to self-dual polygons.

A motivation for this work is the following problem (No 1994-17 in [1]) of V. Arnold:

> Find all projective curves equivalent to their duals. *The answer seems to be unknown even in* $\mathbb{RP}^2$.

(A traditional interpretation of this question would be to consider algebraic curves, in which case the Plücker formulas play a critical role; see [6]. In particular, the Plücker formulas imply that a *nonsingular* self-dual algebraic curve is a conic; however, there exist other, singular, self-dual curves, for example, $y = x^3$, projectively equivalent to its dual $y = x^{3/2}$.)

The main result of this paper is a description of self-dual polygons in $\mathbb{CP}^2$. Let $A_1, A_3, \ldots, A_{2n-1} \in P$ (where the indices are odd residues modulo $2n$) be the vertices of an n-gon, and let $B_2, B_4, \ldots, B_{2n}$ (where the indices are even residues modulo $2n$) be its respective sides: $B_{2i} = A_{2i-1}A_{2i+1}$ for all i. Let m be an odd number, $1 \le m \le n$. The n-gon L is called *m-self-dual* if there exists a projective map $g\colon P \to P^*$ such that $g(A_i) = B^*_{i+m}$ for all i. An example of an m-self-dual n-gon, for arbitrary m, is a regular n-gon. Denote by $\mathcal{M}_{m,n}$ the moduli space of m-self-dual n-gons. Our result is as follows.

Theorem 1 *If* $(m, n) = 1$ *then* $\mathcal{M}_{m,n}$ *consists of one point, the class of a regular n-gon. If* $m < n$, $(m, n) > 1$ *and* $n \neq 2m$ *then* $\dim \mathcal{M}_{m,n} = (m, n) - 1$. *Finally,* $\dim \mathcal{M}_{m,2m} = m - 3$ *and* $\dim \mathcal{M}_{n,n} = n - 3$.

Note, for comparison, that the dimension of the moduli space of n-gons is $2n - 8$. The proof of Theorem 1 occupies Sects. 3 and 4. These sections also contain explicit constructions of self-dual polygons.

The map $g\colon P \to P^*$, associated with an m-self-dual n-gon, determines a linear map $V \to V^*$, defined up to a factor, and therefore a bilinear form F on V. We prove that if the polygon is not a multiple of another polygon then F is symmetric if and only if $m = n$, see Proposition 2. We also show that if an n-self-dual n-gon is convex then the symmetric bilinear form F is definite, Proposition 9.

We make additional observations. First, every pentagon is 5-self-dual, see Proposition 5 below (five is the first interesting number because all triangles are projectively equivalent, and so are all quadrilaterals). Secondly, if an n-gon with odd n is inscribed in a conic and circumscribed about a conic then it is n-self-dual, see Proposition 7. However the moduli space of such "Poncelet" polygons has dimension two, which is less than $n - 3$ for $n \ge 7$.

In the real case, one can also interpret a polygon as a closed polygonal curve. In Sect. 5, we define a polygonal curve as a polygon in $\mathbb{RP}^2$ with two additional structures: every two consecutive vertices A_{2i-1} and A_{2i+1} partition the real projective line B_{2i} into two segments, and one of these segments is chosen (as a side); every two consecutive sides B_{2i} and B_{2i+2}

determine two pairs of vertical angles at the vertex A_{2i+1}, and one of these pairs is chosen (as an exterior angle). Polar duality naturally extends to these polygonal curves, and one can consider self-dual polygonal curves. A given n-gon L gives rise to 2^{2n} polygonal curves. We prove (Proposition 14) that if L is m-self-dual then, out of these 2^{2n} polygonal curves, $2^{(m,n)}$ are m-self-dual.

Sect. 6 concerns self-dual curves and wave fronts in the real projective plane P. We do not attempt to give a complete classification of such curves. A curve $\gamma(t) \subset P$, $t \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, is called self-dual if there exists a projective transformation $g\colon P \to P^*$ and a diffeomorphism φ of S^1 such that $g(\gamma(\varphi(t)) = \gamma^*(t)$. The diffeomorphism φ is a continuous analog of the cyclic shift by m in the definition of m-self-dual n-gons.

A number of results that we establish for polygons have analogs for curves. For example, the bilinear form F is symmetric if and only if $\varphi^2 = \mathrm{id}$, see Proposition 16. If, in addition, a self-dual curve is convex then F is definite, Proposition 17.

We observe that curves of constant width $\pi/2$ on the unit sphere S^2 project to self-dual curves with $\varphi^2 = \mathrm{id}$ in $\mathbb{RP}^2$. We construct such curves of constant width as Legendrian curves in the manifold of contact elements of S^2 satisfying certain monodromy conditions. We also give a similar description to self-dual curves with the diffeomorphism φ having higher order than 2. This description leads to explicit formulas for self-dual curves.

Finally, we briefly discuss the Radon curves, the unit circles in two-dimensional normed spaces for which the orthogonality relation is symmetric. Radon curves have been extensively studied; they provide examples of projectively self-dual curves.

Let us finish this introduction with a question: can a smooth convex self-dual curve, other than a conic, be an oval of an algebraic curve?

2 Polygons and duality

We use the notation from Sect. 1. Let $P = \mathbb{CP}^2$ and $V = \mathbb{C}^3$. Let L be an n-gon in P whose vertices are $A_1, A_3, \dots, A_{2n-1}$ and whose sides are $B_2, B_4, \dots, B_{2n}$. The dual n-gon L^* in the dual projective plane P^* has the vertices $B_2^*, B_4^*, \dots, B_{2n}^*$ and the sides $A_1^*, A_3^*, \dots, A_{2n-1}^*$.

Assume that L is m-self-dual where m is an odd number, $1 \le m \le n$. Then there exists a linear isomorphism $f\colon V \to V^*$ that takes the line $A_i \subset V$ to the line $B_{i+m}^* \subset V^*$ for all i. Along with f, we shall consider the corresponding projective isomorphism $\hat{f}\colon P \to P^*$ and the bilinear form F on V, $F(v, w) = \langle f(v), w\rangle$. Obviously, for a given m-self-dual polygon, $\hat{f}$ is unique, while f and F are unique up to a nonzero constant factor.

Along with an n-gon $L = A_1A_3 \dots A_{2n-1}$, we can consider the kn-gon $kL = A_1A_3 \dots A_{2kn-1}$ with $A_i = A_{i+2n}$. Obviously, $(kL)^* = kL^*$, and if L is m-self-dual, then kL is $(m + 2rn)$-self-dual for $r = 0, 1, \dots, k-1$. A polygon L is called *simple*, if $L \ne kL'$ for any $k > 1$ and L'.

Proposition 2 *Let L be a simple m-self-dual n-gon and $f\colon V \to V^*$ be the corresponding isomorphism. Then f is self-adjoint (or, equivalently, F is symmetric) if and only if $m = n$.*

Proof Notice that

$$F(A_i, A_j) = 0 \iff \langle f(A_i), A_j\rangle = 0 \iff \langle B_{i+m}^*, A_j\rangle = 0$$
$$\iff A_j \in B_{i+m} = A_{i+m-1}A_{i+m+1}.$$

In particular, $F(A_i, A_{i+m\pm 1}) = 0$ for all i.

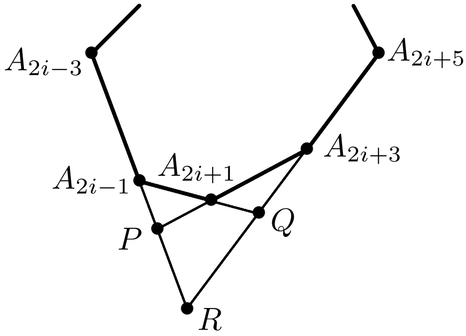

Fig. 1 Cross-ratios at the vertices of a polygon

Let F be symmetric. Then, for all i, $F(A_{i+m\pm1}, A_i) = 0$, and hence

$$A_i \in A_{(i+m-1)+(m-1)}A_{(i+m-1)+(m+1)} = A_{i+2m-2}A_{i+2m},$$
$$A_i \in A_{(i+m+1)+(m-1)}A_{(i+m+1)+(m+1)} = A_{i+2m}A_{i+2m+2}.$$

Since the points $A_{i+2m-2}, A_{i+2m}, A_{i+2m+2}$ are not collinear, this means that $A_i = A_{i+2m}$. Hence $m = n$ (the polygon L is simple!).

Let $m = n$. For every i, $F(A_i, A_{i+m\pm1}) = 0$, and, in addition to that,

$$F(A_{i+m-1}, A_i) = F(A_{i+m-1}, A_{i+2m}) = F(A_{i+m-1}, A_{(i+m-1)+(m+1)}) = 0,$$
$$F(A_{i+m+1}, A_i) = F(A_{i+m+1}, A_{i+2m}) = F(A_{i+m+1}, A_{(i+m+1)+(m-1)}) = 0.$$

This implies that the linear forms $F(A_i, \cdot)$ and $F(\cdot, A_i)$ are proportional for every i, that is, there exist non-zero complex numbers λ_i such that $F(A_i, x) = \lambda_i F(x, A_i)$ for all i and x. Hence $F(A_i, A_j) = \lambda_i F(A_j, A_i) = \lambda_i\lambda_j F(A_i, A_j)$, so if $F(A_i, A_j) \neq 0$, then $\lambda_i = \lambda_j^{-1}$. But $F(A_i, A_{i+m-3}) \neq 0$ (because A_{i+m-3} does not belong to the line $A_{i+m-1}A_{i+m+1}$, which is the zero locus of the form $F(A_i, x)$). Hence

$$\lambda_i = \lambda_{i+m-3}^{-1} = \lambda_{i+2(m-3)} = \lambda_{i+3(m-3)}^{-1} = \cdots = \lambda_{i+m(m-3)}^{-1} = \lambda_i^{-1},$$

so $\lambda_i = \pm1$. We state that all λ_i's are the same. Indeed, if $\lambda_i = 1$, $\lambda_j = -1$ for some i, j, then for every k, one of $F(A_i, A_k)$, $F(A_j, A_k)$ had to be 0, that is, A_k would belong to one of two lines, $A_{i+m-1}A_{i+m+1}$ and $A_{j+m-1}A_{j+m+1}$, that is, all vertices of the polygon would belong to two lines, which is impossible, since the three lines A_1A_2, A_2A_3, A_3A_4 are all different. We see that our form F is either symmetric or skew symmetric. However it cannot be skew-symmetric because it is non-degenerate, and all skew-symmetric forms in an odd-dimensional space are degenerate. □

Remark 3 The class of projective equivalence of an n-gon L is determined by a collection of $2n$ numbers $(p_1, q_1, p_3, q_3, \ldots, p_{2n-1}, q_{2n-1})$ (the indices are odd residues modulo $2n$). The definition of these numbers refers to Fig. 1:

$$p_{2i+1} = [A_{2i-3}, A_{2i-1}, P, R], \quad q_{2i+1} = [R, Q, A_{2i+3}, A_{2i+5}]$$

where $P = B_{2i-2} \cap B_{2i+2}$, $R = B_{2i-2} \cap B_{2i+4}$, $Q = B_{2i} \cap B_{2i+4}$ and $[\cdot,\cdot,\cdot,\cdot]$ denotes the cross-ratio of four points on a projective line, see [9, 11]. These $2n$ numbers are not independent: they satisfy 8 relations ensuring that the polygon is closed. Similarly, the dual polygon L^* is characterized by the respective cross-ratios $(p_2^*, q_2^*, \ldots, p_{2n}^*, q_{2n}^*)$ (the indices are even residues modulo $2n$). It is easy to see that $p_{2i}^* = q_{2i-1}$ and $q_{2i}^* = p_{2i+1}$. An n-gon L is m-self-dual if and only if $p_i = p_{i+m}^*$, $q_i = q_{i+m}^*$, and hence if and only if $p_i = q_{i+m-1}$, $q_i = p_{i+m+1}$. This implies $2m$-periodicity of the sequence of cross-ratios: $p_i = p_{i+2m}$, $q_i = q_{i+2m}$, cf. Sect. 4.

3 The case $m = n$

In this section, we consider n-gons with odd n and with every vertex dual to the opposite side of a projectively equivalent n-gon. According to Proposition 2, the bilinear form F is symmetric in this case, so it determines a (complex) Euclidean structure in the space V. In an appropriate coordinate system, the projective duality becomes the *polar duality*

$$(a, b, c) \mapsto \{ax + by + cz = 0\}.$$

(Geometrically, this means that we apply to a point of the standard Euclidean plane the inversion in the unit circle centered at 0, then reflect the point in 0, and then take a line through the obtained point perpendicular to the position vector of this point.) For this duality, we will use the notations $A \mapsto A^\perp \mapsto (A^\perp)^\perp = A$. Two polygons, n-self-dual with respect to this duality, are projectively equivalent if and only if they are $O(3, \mathbb{C})$-equivalent.

For a polygon $A_1 A_3 \ldots A_{2n-1}$, construct the star-like polygon $C_1 C_2 \ldots C_n$ where $C_i = A_{1+(i-1)(n-1)}$.

Lemma 4 *The polygon $A_1 A_3 \ldots A_{2n-1}$ is n-self-dual (with respect to the polar duality) if and only if $C_{i+1} \in C_i^\perp$ for all i (with i a residue modulo n).*

Proof Obvious. □

This leads to a simple explicit construction of all n-self-dual n-gons. Fix a point C_1, then a point $C_2 \in C_1^\perp$ not equal to C_1 (the latter is relevant only if $C_1 \in C_1^\perp$). Notice that, modulo the action of $O(3, \mathbb{C})$, there are four choices of the pair C_1, C_2 (depending on possible incidences $C_1 \in C_1^\perp$, $C_2 \in C_2^\perp$). Then choose $C_i \in C_{i-1}^\perp$, $C_i \neq C_{i-2}, C_{i-1}$ for $i = 2, \ldots, n-1$, with the additional requirement $C_{n-1} \neq C_1$. In conclusion, we put $C_n = C_{n-1}^\perp \cap C_1^\perp$. After this, we redenote the points, $C_i = A_{1+(i-1)(n-1)}$, and get an n-self-dual n-gon $A_1 A_3 \ldots A_{2n-1}$. Moreover, up to a projective equivalence preserving the numeration of vertices, this construction gives all n-self-dual n-gons, one time each.

In particular, the moduli space of n-self-dual n-gons has dimension $n - 3$ (each of the points $C_3, \ldots, C_{n-1}$ is arbitrarily chosen within a line with finitely many punctures).

Pentagons For an arbitrary $n \geq 4$, the moduli space of all n-gons has dimension $2n - 8$ ($2n$ for n vertices, -8 for the action of the group $PSL(3, \mathbb{C})$). In general, this exceeds the dimension $n - 3$ of n-self-dual n-gons, but for $n = 5$ the two numbers coincide: both equal 2. Moreover, the following holds.

Proposition 5 *Every pentagon is 5-self-dual.*

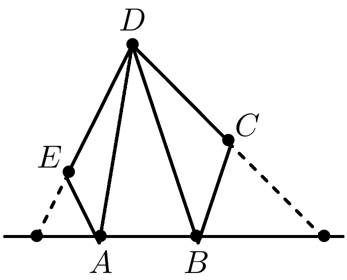

Fig. 2 Pentagons are self-dual

Proof We will prove this for a "generic" pentagon with no three vertices collinear; the general case can be resolved by a transition to limit. For a pentagon $ABCDE$ there are 5 cross-ratios $\rho_A, \dots, \rho_E$: ρ_A is defined as the cross-ratio of the lines AB, AC, AD, AE, and the other four are defined in a similar way. These cross-ratios projectively determine a pentagon: the points A, B, C, D can be moved to chosen locations, and after that the lines BE and CE are determined by ρ_B and ρ_C. (Certainly, the five cross-ratios are not independent: generally, two of them determine the rest.) For the dual pentagon, these cross-ratos are $\rho_{AB}, \dots, \rho_{AE}$ where ρ_{AB} is the cross-ratio of the points $DE \cap AB$, A, B, $DC \cap AB$ on the line AB, and the other four are defined in a similar way. Fig. 2 shows that $\rho_E = \rho_{BC}$, and four similar equalities hold as well. □

Poncelet polygons We begin with the following easy statement.

Lemma 6 *Let $C \subset P$ be a nondegenerate conic, and let $L = E_1 \dots E_n$ be an n-gon inscribed in C. Then the n-gon whose sides are tangent to C at points $E_1, \dots, E_n$ is (projectively equivalent to the) dual to L. More precisely, there exists a projective isomorphism $P \to P^*$ that takes E_i to the tangent line to C at E_i.*

Proof Since all nondegenerate conics are projectively equivalent, we may assume that C is a unit circle in the Euclidean plane. Let E'_i be the point of C opposite to E_i. Then the polygon L is projectively equivalent to the polygon $L' = E'_1 \dots E'_n$ and the tangent to C at E'_i is polar dual to E_i. □

An n-gon with odd n is called a *Poncelet polygon* if it is both inscribed into a nondegenerate conic and circumscribed about a nondegenerate conic (see Fig. 3).

Proposition 7 *Every Poncelet n-gon is n-self-dual.*

Proof This follows from Lemma 6 and the following known result [7, 10]. Let $A_1A_3 \dots A_{2n-1}$ be an n-gon (with odd n) inscribed into a conic C and circumscribed about a conic C'. Then there exists a projective involution $h: P \to P$ such that $h(A_i)$ is the tangency point of $A_{i+n-1}A_{i+n+1}$ and C'. □

Two more remarks. The first is that any non-degenerate pentagon is a Poncelet polygon, so Proposition 5 follows from Proposition 7. The second is the following proposition.

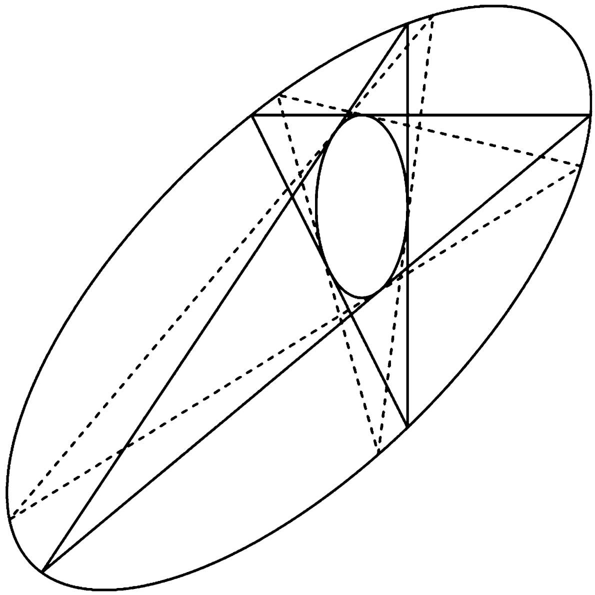

Fig. 3 Poncelet pentagons

Proposition 8 *For every odd $n \geq 5$, the projective moduli space of Poncelet n-gons is two-dimensional.*

Proof The conics C, C' from the definition of Poncelet polygons determine a one-parameter family $\mathcal{F}$ of conics that have four common tangents. Generically, there exists a unique, up to a projective equivalence, such family $\mathcal{F}$ (to specify this family, it suffices to fix a generic quadruple of lines). For every $C \in \mathcal{F}$, there exists a finite number of $C' \in \mathcal{F}$ such that some n-gon inscribed in C is circumsribed about C' (see [5] for an explicit condition due to Cayley). Moreover, for such a pair C, C', every point of C is a vertex of such an n-gon (Poncelet's theorem, see [3]). Thus, a projective class of a Poncelet n-gon is determined by two independent choices: the choice of a $C \in \mathcal{F}$ and the choice of a point in C. □

Thus, for an odd $n > 5$, Poncelet n-gons form a small fraction of the space of n-self-dual n-gons.

The real Euclidean case Let L be a real n-self-dual n-gon. Then F is a real symmetric bilinear form (determined up to real non-zero factors), and there are two possibilities: the form F may be definite or indefinite. In the definite (Euclidean) case, the construction of a self-dual polygon given in the beginning of this section looks especially simple. Consider the unit sphere $S \subset \mathbb{R}^3$. Choose an arbitrary point $C_1 \in S$. Then choose a point C_2 at the distance $\pi/2$ from C_1. Then choose a point C_3 at the distance $\pi/2$ from C_2, not equal to $\pm C_1$, then choose $C_4, C_5, \ldots$. The last choice will be slightly different from the preceding ones: we choose the point C_{n-1} at the distance $\pi/2$ from C_{n-2}, not equal to $\pm C_{n-3}$, and also not equal to $\pm C_1$. After this, we denote by C_n a point at the distance $\pi/2$ from each of the points C_{n-1} and C_1. (There are two such points, they form the intersection of two different great circles.) Then we put $C_i = A_{1+(i-1)(n-1)}$ and project the polygon $A_1 A_3 \ldots A_{2n-1}$ onto P. This is our self-dual polygon (see Fig. 4).

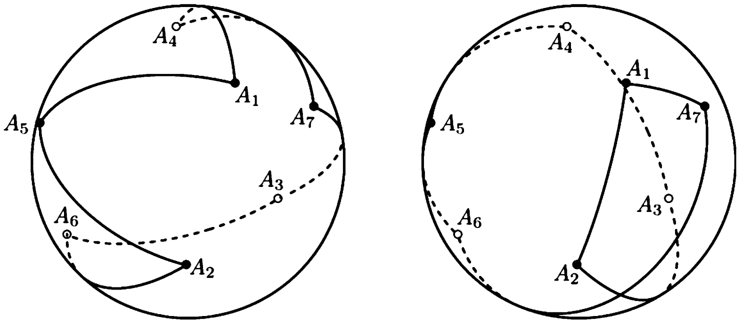

Fig. 4 Self-dual polygon on the unit sphere

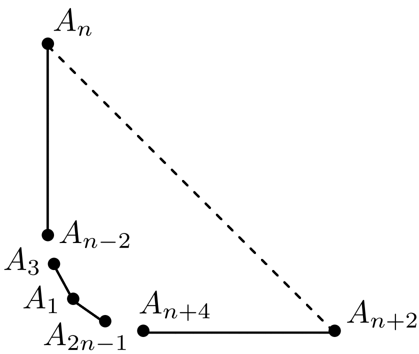

Fig. 5 Convex heptagon: proof of Proposition 9

It is natural to ask which n-self-dual n-gons correspond to definite forms. A partial answer to this question is provided by the following proposition.

Proposition 9 *If a real n-self-dual n-gon is projectively equivalent to an affine convex n-gon, then the corresponding symmetric bilinear form is definite.*

Proof Let $A_1A_3\dots A_{2n-1}$ be our convex polygon. We will use only the convexity of the heptagon $A_1A_3A_{n-2}A_nA_{n+2}A_{n+4}A_{2n-1}$. We will assume that the points A_{n-2}, A_n, A_{n+2}, A_{n+4} have projective coordinates $(0:1:1)$, $(0:1:0)$, $(1:0:0)$, $(1:0:1)$. Then the lines $B_{n-1}=A_{n-2}A_n$, $B_{n+1}=A_nA_{n+2}$, $B_{n+3}=A_{n+2}A_{n+4}$ are, respectively, the y-axis, the line at infinity, and the x-axis (see Fig. 5). Therefore $B^*_{n-1}=(1,0,0)$, $B^*_{n+1}=(0,0,1)$, $B^*_{n+3}=(0,1,0)$. The matrix of the isomorphism $G=F^{-1}\colon P^*\to P$ which takes B^*_j into A_{j+n} is symmetric by Proposition 2. Let it be

$$G=\begin{bmatrix} a & b & c\\ b & d & e\\ c & e & f\end{bmatrix}.$$

Then $A_{2n-1} = G(B^*_{n-1}) = (a, b, c)$, $A_1 = G(B^*_{n+1}) = (c, e, f)$, $A_3 = G(B^*_{n+3}) = (b, d, e)$, and the affine coordinates of the points A_{2n-1}, A_1, A_3 are

$$x_{2n-1} = \frac{a}{c}, \quad y_{2n-1} = \frac{b}{c}, \quad x_1 = \frac{c}{f}, \quad y_1 = \frac{e}{f}, \quad x_3 = \frac{b}{e}, \quad y_3 = \frac{d}{e}.$$

The conditions of convexity of our heptagon are

$$0 < x_3 < x_1 < x_{2n-1} < 1, \quad 1 > y_3 > y_1 > y_{2n-1} > 0$$
$$\text{and} \quad \det \begin{bmatrix} x_1 - x_{2n-1} & y_1 - y_{2n-1} \\ x_1 - x_3 & y_1 - y_3 \end{bmatrix} > 0;$$

the latter means

$$x_1 y_{2n-1} + x_3 y_1 + x_{2n-1} y_3 - x_1 y_3 - x_3 y_{2n-1} - x_{2n-1} y_1 > 0,$$

or

$$2 \cdot \frac{b}{f} + \frac{a}{c} \cdot \frac{d}{e} - \left(\frac{c}{f} \cdot \frac{d}{e} + \frac{b}{e} \cdot bc + \frac{a}{c} \cdot \frac{e}{f} \right) > 0.$$

After multiplication by the positive number $c^2 f^2 x_{2n-1} y_1 = acef$, this inequality becomes

$$2acbe + a^2 df - (ac^2 d + ab^2 f + a^2 e^2) > 0,$$

that is, $a \cdot \det G > 0$. Also it follows from the convexity inequality that $ad - b^2 = f^2 x_1 y_1 (x_{2n-1} y_3 - x_3 y_{2n-1}) > 0$. These two inequalities show that the form F is definite. □

It should be noted that, as it is seen from the construction above (with a sequence of points on the sphere), a self-dual polygon with a definite form does not need to be convex. Still, it is true that a pentagon has a definite form if and only if it is projectively equivalent to a convex pentagon (we leave a proof to the reader).

4 The case $m < n$

Let $L = A_1 A_3 \dots A_{2n-1}$ be an m-self-dual polygon with $m < n$. By Proposition 2, the form F in this case is not symmetric. If B is a point or a line in P, then $B^\perp$ is defined as $\{y \in P \mid F(y, x) = 0 \text{ for all } x \in B\}$. There arises a projective transformation $G\colon P \to P$, $G(B) = (B^\perp)^\perp$, and obviously, in term of matrices, $G = F^{-1} F^t$.

Lemma 10 *If L is m-self-dual then $G(A_i) = A_{i+2m}$.*

Proof One has $A_i^\perp = A_{i+m-1} A_{i+m+1}$, and hence $G(A_i) = (A_i^\perp)^\perp = A_{i+m-1}^\perp \cap A_{i+m+1}^\perp = (A_{i+2m-2} A_{i+2m}) \cap (A_{i+2m} A_{i+2m+2}) = A_{i+2m}$. □

Thus, G makes a nontrivial cyclic permutation of vertices of L, and, in particular, $G^r = \mathrm{id}$ where $r = \frac{n}{(m,n)}$. The following result is an elementary fact from linear algebra.

Lemma 11 *Let F be a nondegenerate nonsymmetric bilinear form in V. Then there exists a basis in V with respect to which F has one of the following matrices*:

$$H_\varphi = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Proof There is a unique decomposition $F = F_+ + F_-$ where F_+ is symmetric and F_- is skew-symmetric. Let $W = \operatorname{Ker} F_-$; since $F_- \neq 0$, $\dim W = 1$.

Case 1 $F_+|_W \neq 0$, $\operatorname{rank} F_+ = 3$. Let $e_3 \in W$, $F_+(e_3, e_3) = 1$. Let Z be the orthogonal complement to W with respect to F_+. Choose $e'_1, e'_2 \in Z$ with $F_+(e'_i, e'_j) = \delta_{ij}$. Then choose a rescaling $e_1 = \alpha e'_1$, $e_2 = \alpha e'_2$ such that $F_+(e_1, e_1)^2 + F_-(e_1, e_2)^2 = 1$. (Notice that $F_+(e'_1, e'_1)^2 + F_-(e'_1, e'_2)^2 = \det F \neq 0$.) Then the matrix of F with respect to the basis e_1, e_2, e_3 is H_φ with some (complex) $\varphi \neq k\pi/2$.

Case 2 $F_+|_W \neq 0$, $\operatorname{rank} F_+ = 2$. Let e_3, Z denote the same as in Case 1, let $0 \neq e'_2 \in \operatorname{Ker} F_+$, and let $e_1 \in Z - \operatorname{Ker} F_+$, $F_+(e_1, e_1) = 1$. Choose α such that, for $e_2 = \alpha e'_2$, $F_-(e_1, e_2) = 1$. Then the matrix of F with respect to the basis e_1, e_2, e_3 is J.

Case 3 $F_+|_W \neq 0$, $\operatorname{rank} F_+ = 1$. Let e_3 denote the same as in Cases 1 and 2, and let e_1, e_2 be a basis in $\operatorname{Ker} F_+$ such that $F_-(e_1, e_2) = 1$. Then the matrix of F with respect to the basis e_1, e_2, e_3 is

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{which is } H_{\pi/2}.$$

Case 4 $F_+|_W = 0$, $\operatorname{rank} F_+ = 3$. Choose a nonzero vector $e'_3 \in W$. Let Z be the orthogonal complement to W, and let C be the "light cone" $\{x \in V \mid F_+(x, x) = 0\}$. Since $C \not\subset Z$, we can choose an $e'_2 \in V$ such that $F_+(e'_2, e'_2) = 0$, $F_+(e'_2, e'_3) = 1$. Let $U \in V$ be the subspace spanned by e'_2, and Y be the orthogonal complement of U. The intersection $Y \cap Z$ is not contained in $W + U$: if a linear combination of e'_2 and e'_3 is orthogonal to both e'_2 and e'_3, then it must be 0. Take $e_1 \in Y \cap Z$ with $F_+(e_1, e_1) = 1$. Then $F_-(e_1, e'_2) \neq 0$ (otherwise F_- would have been zero). Put $e_2 = \alpha e'_2$, $e_3 = \alpha^{-1} e'_3$ in such a way that $F_-(e_1, e_2) = 1$. Then the matrix of F with respect to the basis e_1, e_2, e_3 is K.

Case 5 $F_+|_W = 0$, $\operatorname{rank} F_+ < 3$. Take a 1-dimensional space $U \subset \operatorname{Ker} F_+$. If $U = W$, then $\operatorname{Ker} F \supset W$ is nonzero, so F is degenerate. If $U \neq W$, then both F_+ and F_- are zero on $U \oplus W$, which also means that F is degenerate. □

Return now to the bilinear form F related to our m-self-dual n-gon L.

Proposition 12 *In an appropriate coordinate system, the matrix of F is H_φ with $r\varphi \in \pi\mathbb{Z}$. Moreover, if the n-gon L is simple, then $s\varphi \notin \pi\mathbb{Z}$ for any positive $s < r$.*

Proof According to Lemma 11, the matrix of F, in an appropriate basis, is H_φ, J, or K. But

$$J^{-1}J^t = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K^{-1}K^t = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

and neither of these two matrices has finite order (for both, the Jordan form contains a nontrivial Jordan block). On the other hand, the matrix of G is $(H_\varphi)^{-1}H_\varphi^t = H_{-\varphi}^2 = H_{-2\varphi}$, that is (again, in an appropriate coordinate system), $H_{-2\varphi}A_i = A_{i+2m}$. First, this shows that $H_{-2\varphi}^r = I$, that is, $2r\varphi$ is a multiple of 2π. Second, if $2s\varphi$ is a multiple of 2π for a positive $s < r$, then $A_i = A_{i+2sm}$ where sm is not a multiple of n, so our polygon is not simple. □

Thus, L contains (m,n) regular r-gons, $A_iA_{i+2m}\dots A_{i+(r-1)m}$, $i = 1, 2, \dots, (m,n)$. [By a regular n-gon we understand a (possibly self-intersecting) n-gon in the Euclidean plane with all lengths of the sides equal and all angles equal; thus, there are two projectively different types of regular pentagons, three projectively different types of regular heptagons, and so on.] If $(m,n) = 1$, then L itself is regular (so, in this case, an m-self-dual n-gon is projectively unique). If $(m,n) > 1$, then this uniqueness, in general, does not hold. Below, we give an explicit construction of all m-self-dual n-gons which will demonstrate this non-uniqueness.

First, notice that, in our case, the projective duality has a simple geometric description: we consider a Euclidean plane (with a fixed origin) and, for a point A ($\neq 0$), the dual line A^* is obtained from the polar dual $A^\perp$ by a clockwise rotation about the origin by the angle $\pi(n-m)/n$.

Now, let us construct an arbitrary m-self-dual n-gon. In addition to $r = \frac{n}{(m,n)}$, put $k = \frac{m}{(m,n)}$ and also $d = (m,n)$; thus, $mr = kn$. In the Euclidean plane with a fixed origin O, choose an arbitrary point A_1. Then successive counter-clockwise rotations by the angle $2\pi m/n$ give the points $A_{2m+1}, A_{4m+1}, \dots, A_{2(r-1)m+1}$, and also the lines $A_1^* = A_mA_{m+2}$, $A_{2m+1}^* = A_{3m}A_{3m+2}$, $A_{4m+1}^* = A_{5m}A_{5m+2}$, $\dots$, $A_{2(r-1)m+1)}^* = A_{(2r-1)m}A_{(2r-1)m+2}$. Of the numbers $m, 3m, 5m, \dots, 2(r-1)m$, one is d modulo $2n$ ($d = um + v\cdot 2n$ for a unique u, $0 \le u < 2r$, and this u must be odd). So, one of our lines should be A_dA_{d+2}; choose a point A_d on this line. This choice gives also the points $A_{2m+d}, A_{4m+d}, \dots, A_{2(r-1)m+d}$ and the lines $A_{d+m-1}A_{d+m+1}$, $A_{d+3m-1}A_{d+3m+1}$, $A_{d+5m-1}A_{d+5m+1}$, $\dots$, $A_{d+(2r-1)m-1}A_{d+(2r-1)m+1}$. By the way, one of these lines will be $A_{2n-1}A_1$. Our next choice will be a point A_{d+2}, again on the line A_dA_{d+2}. This will give us r additional points (including A_{d+2}) and r lines, dual to these points. One of these lines will be A_1A_3, and we choose a point A_3 on it. One of the lines coming with this point will be $A_{d+2}A_{d+4}$, and we choose a point A_{d+4}, and so on. Proceeding in this way, we choose the points in the following order: A_1, A_d, A_{d+2}, A_3, A_{d+4}, A_5, A_{d+6}, A_7, $\dots$, A_{2d-3}, A_{d-2}. Here we stop: the next choice should be A_{2d-1}, but this point will appear as the intersection of the line $A_{2d-1}A_{2d+1}$ coming with the point A_d and the line $A_{2d-3}A_{2d-1}$ coming with the point A_{d-2}. After that, we have the points $A_1, A_3, A_5, \dots, A_{2d-1}$, and hence we have all the vertices of our polygon.

The projective symmetry G, $G(A_i) = A_{i+2m}$, shows that every m-self-dual n-gon is also m'-self-dual for every $m' \equiv em \bmod n$ where e is odd. In particular, if $(e,r) = 1$, then m-self-dual n-gons and m'-self-dual n-gons are the same n-gons (although their self-dualities involve different projective isomorphisms $P \to P^*$). On the other hand, if n is odd, then we see that every m-self-dual n-gon is also n-self-dual, that is, it belongs to the class of polygons considered in Sect. 3.

Fig. 6 shows two 3-self-dual 9-gons. As was remarked above, they are also 9-self-dual (with respect to the polar duality). Note that no n-gon with n even is n-self-dual (the definition of m-self-duality requires that m is odd); but they must be centrally symmetric (with respect to the affine chart considered in this section), and the 12-gons of Fig. 7 are centrally symmetric indeed.

Proposition 13 *Let $m < n$ and $(m,n) > 1$. Then the moduli space of m-self-dual n-gons has dimension $(m,n) - 1$, for $n \neq 2m$, and $(m,n) - 3$, for $n = 2m$.*

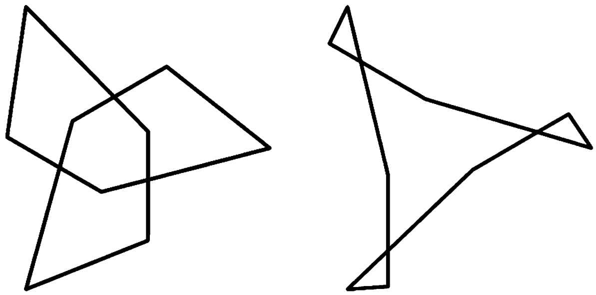

Fig. 6 Two 3-self-dual nonagons

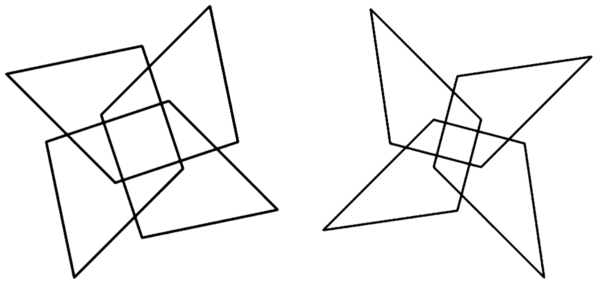

Fig. 7 Two 3-self-dual dodecagons

Proof If the basis, in which the bilinear form F has the canonical form H_φ, is chosen then, to specify our m-self-dual n-gon (with $m < n$), we need to choose a point A_1 in the plane (minus one point), which depends on two parameters, and then $d-2$, $d=(m,n)$, points $A_d, A_{d+2}, A_3, A_{d+4}, A_5, \dots, A_{d-2}$ which provide $(m,n)-2$ more parameters, with the total of (m,n). From this number, we need to subtract the number of parameters on which the basis for a given form F depends (in other words, the dimension of the Lie group of linear transformations of V that preserve the form H_φ). If 2φ is not a multiple of π, which corresponds to the case $n \neq 2m$, this dimension is 1. This is seen from Case 1 of the proof of Lemma 11: the choice of e_3 provides no parameters (it is two-valued), then we choose e_1' and e_2' on the conic $F_+(x,x)=0$ with the condition $F_+(e_1',e_2')=0$, which provides one parameter, and then we multiply both e_1' and e_2' by the same complex number which we determine from a quadratic equation. So the total number of free parameters in this case is 1, and the dimension of the moduli space is $(m,n)-1$. If $n=2m$, then $\cos\varphi=0$, and this is Case 3 of the proof of Lemma 11. In this case, the choice of e_3 does not provide any parameters while e_1 and e_2 are chosen up to the action of the group of transformations of Z preserving the form F_-; it is $SL(2;\mathbb{C})$, the dimension is 3. Thus, if $n=2m$, then the dimension of the moduli space is $(m,n)-3=m-3$. □

Notice in conclusion that our results show that the space of moduli of m-self-dual hexagons has dimension 0, whatever m is. Actually, the only self-dual hexagon (for any m) is the regular hexagon; again, we leave the details to the reader.

5 Polygonal curves

A *polygonal curve* is a real polygon $A_1A_3\dots A_{2n-1}$ with the following two (independent) additional structures. (1) For every even i, one of the two segments into which the points $A_{i-1},\ A_{i+1}$ cut the real projective line B_i is chosen; we will refer to this segment as an *edge* of the polygonal curve. (2) For every odd i, one of the two pairs of vertical angles formed by the real projective lines $B_{i-1},\ B_{i+1}$ is fixed; we will refer to these angles as *exterior angles* of the polygonal curve. Thus, every real n-gon gives rise to 2^{2n} polygonal curves.

For a polygonal curve $A_1A_3\dots A_{2n-1}$ (with its additional structures), there arises a *dual polygonal curve* in dual real projective plane P^*. This is the polygon $B_2^*B_4^*\dots B_{2n}^*$ with the following edges and exterior angles. The edge $B_{2i}^*B_{2i+2}^*$ is formed by points of P^* dual to the lines through A_{2i+1} in P contained in the exterior angles at A_{2i+1} of the given polygonal curve. The exterior angles at B_{2i}^* is formed by lines in P^* dual to the points of the edge $A_{2i-1}A_{2i+1}$ of the given polygonal curve.

Let $m\le n$ be a positive odd number. A polygonal curve $A_1A_3\dots A_{2n-1}$ is called m-self-dual if there exists a projective isomorphism $P\to P^*$ which takes A_i into B_{i+m}^* and also takes edges and exterior angles of the polygonal curve $A_1A_3\dots A_{2n-1}$ into edges and exterior angles of the polygonal curve $B_2^*B_4^*\dots B_{2n}^*$.

Proposition 14 *Let $L=A_1A_3\dots A_{2n-1}$ be an m-self-dual n-gon. Then, of the 2^{2n} polygonal curves arising from the polygon L, $2^{(m,n)}$ are m-self-dual.*

Proof Choose edges $A_1A_3,\ A_3A_5,\dots,\ A_{2d-1}A_{2d+1}$ in an arbitrary way. If $m<n$, then apply to these edges $r=n/d$ consecutive rotations by the angle $2\pi m/n$; we will get a full set of edges. Then apply the duality to these edges, and this gives a choice of exterior angles for the dual polygon L^*. The projective isomorphism between L and L^* makes these angles exterior angles for L, and L becomes an m-self-dual polygonal curve. Obviously, this construction gives all m-self-dual n-gonal curves. □

Polygonal curves are natural polygonal counterparts to smooth curves. Let us describe a unifying point of view.

The space of contact elements and projective duality A contact element of the real projective plane is a pair (A,B) where $A\in P$ is a point, $B\subset P$ is a line, and $A\in B$. Denote the space of contact elements by F (it is naturally identified with the space of full flags in $\mathbb{R}^3$). One has two projections $\pi_1: F\to P$ and $\pi_2: F\to P^*$ defined by the formulas: $\pi_1(A,B)=A$, $\pi_2(A,B)=B$. The space F has a contact structure (a non-integrable two-dimensional distribution) defined by the condition that the velocity of point A lies in the line B. The fibers of the projections $\pi_{1,2}$ are Legendrian curves (curves tangent to the contact distribution). The space of contact elements of the dual plane P^* is canonically identified with F.

Projective duality is easily described in terms of the space of contact elements. Let $\gamma\subset P$ be a smooth curve. Assigning the tangent line to each point of γ gives a lift $\Gamma\subset F$; this lifted curve is Legendrian. The curve $\pi_2(\Gamma)$ is the dual curve $\gamma^*\subset P^*$, and the lift of γ^* to F is again Γ. A wave front in P is defined as the π_1-projection of a smooth Legendrian curve

Γ in F to P; it has singularities (generically, semi-cubical cusps) at the points where Γ is tangent to the fibers of π_1. The dual wave front is $\pi_2(\Gamma)$.

Likewise for polygonal curves. The set of edges of a polygonal curve provides a closed curve in P, but its lift to F consists of disjoint arcs of the fibers of π_2; to connect these arcs by segments of the fibers of π_1, we need to choose exterior angles. Thus, for an n-gonal curve L in P, its lift to F is a $2n$-gonal curve, whose sides are segments of the fibers of the alternating projections π_1 and π_2, and whose projection to P^* is the dual polygonal curve L^*.

6 Self-dual curves

Many results from the preceding sections extend to self-dual wave fronts in $\mathbb{RP}^2$. Here we do not attempt to give a complete classification of such fronts; instead we describe several classes of examples, including explicit formulas for self-dual curves and fronts. Self-dual curves will be described as Legendrian curves in certain three-dimensional contact manifolds satisfying certain monodromy conditions.

In this section, $P = \mathbb{RP}^2$ and $V = \mathbb{R}^3$. Let $\gamma(t) \subset P$ be a self-dual parameterized closed curve (possibly, with cusps); we assume that the parameter t takes values in $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. The projectively dual curve also has a parameterization, $\gamma^*(t)$: the covector $\gamma^*(t)$, defined up to a nonzero multiplier, vanishes on the vectors $\gamma(t)$ and $\gamma'(t)$. As in Sect. 2, we have a linear isomorphism $f\colon V \to V^*$ that takes the line $\gamma(t) \subset V$ to the line $\gamma^*(\varphi^{-1}(t)) \subset V^*$ where φ is a diffeomorphism of S^1. This diffeomorphism plays the role of the cyclic shift by m in the definition of m-self-dual polygons. As in Sect. 2, we consider the corresponding projective isomorphism $\hat{f}\colon P \to P^*$ and the bilinear form F on V, $F(v,w) = \langle f(v), w\rangle$. As in Sect. 4, we consider the projective transformation $G = (\hat{f})^{-1}\hat{f}^*\colon P \to P$, $G(B) = (B^\perp)^\perp$.

The next lemma is an analog of Lemma 10.

Lemma 15 *One has $G(\gamma(t)) = \gamma(\varphi^2(t))$ for all t.*

Proof The proof is essentially the same as that of Lemma 10. First note that the line $\{y \in P \mid \langle y, \gamma(t)\rangle = 0\}$ is the tangent line $T_{\gamma^*(t)}\gamma^*$ to γ^* at the point $\gamma^*(t)$. It follows that $\gamma(t)^\perp = (\hat{f})^{-1}(T_{\gamma^*(t)}\gamma^*) = T_{\gamma(\varphi(t))}\gamma$. Likewise, the point $\{y \in P \mid \langle y, T_{\gamma(\varphi(t))}\gamma\rangle = 0\}$ is $\gamma^*(\varphi(t))$, and therefore $(\gamma(t)^\perp)^\perp = \gamma(\varphi^2(t))$. □

Analogs of the polygons considered in Sect. 3 are the curves for which the diffeomorphism φ is an involution. Just like a polygon, a curve may be a multiple of another curve. The next proposition is an analog of Proposition 2.

Proposition 16 *Assume that a self-dual curve γ is not a multiple of another curve. Then $\varphi^2 = \mathrm{id}$ if and only if the bilinear form F is symmetric.*

Proof The form F is symmetric if and only if G is the identity. If $\varphi^2 = \mathrm{id}$ then, by Lemma 15, $G(\gamma(t)) = \gamma(t)$ for all t. Since γ contains four points in general position, $G = \mathrm{id}$. Conversely, if $G = \mathrm{id}$ then $\gamma(t) = \gamma(\varphi^2(t))$ for all t. Since γ is not a multiple of another curve, $\varphi^2 = \mathrm{id}$. □

An analog of Proposition 9 holds as well.

Proposition 17 *If γ is a convex self-dual curve such that $\varphi^2 = \mathrm{id}$ then the symmetric bilinear form F is definite.*

Proof Assume it is not. Then, in an appropriate coordinate system, the respective quadratic form is $x^2 + y^2 - z^2$. The light cone $x^2 + y^2 = z^2$ projects to a circle $C \subset \mathbb{RP}^2$. If $p \in C$ then the line $p^\perp$ is the tangent line to C at point p, and if a point p is inside C then the line $p^\perp$ lies in the exterior of C.

Due to the convexity, $\gamma(t) \notin \gamma(t)^\perp$ for all t, hence γ does not intersect C. Therefore, γ lies either inside C or outside of it. In the former case, the envelope γ^* of the lines $\gamma(t)^\perp$ lies outside of C and cannot coincide with γ. In the later case, one can find a tangent line ℓ to γ, disjoint from C; then the point $\ell^\perp \in \gamma^*$ lies inside C, and again γ fails to coincide with γ^*. □

Spherical curves Let us consider the case when the symmetric bilinear form F is (positive) definite. Then the correspondence between points and the dual lines is that between pairs of antipodal poles and the corresponding equators on the unit sphere (that doubly covers $\mathbb{RP}^2$). Thus the projective duality moves every point of a curve $\gamma \subset S^2$ distance $\pi/2$ in the normal direction to γ.

An example of a self-dual curve on S^2 is a circle of radius $\pi/4$. This circle is included into the family of curves of constant width $\pi/2$ that are all self-dual (of course, all distances are in the spherical metric). We interpret curves of constant width as Legendrian curves.

Let M be the space of oriented geodesic segments of length $\pi/2$ on S^2. Then M is diffeomorphic to $SO(3) \cong \mathbb{RP}^3$. Define a two-dimensional distribution E on M by the condition that the velocities of the end points of a segment are perpendicular to the segment. Assign an oriented contact element to a geodesic segment AB: the foot point is the midpoint of AB and the direction is the oriented normal to AB. This provides an identification of M with the space of oriented contact elements F of S^2.

Lemma 18 *Under the diffeomorphism $F \cong M$, the standard contact structure in F is identified with the distribution E.*

Proof The space E is generated by two vector fields corresponding to the following motions of a geodesic segment AB: the rotation of AB about its midpoint, and the rotation of AB about the axis AB (so that the end points are fixed). The velocities of the corresponding motions of the midpoint of AB are orthogonal to AB, which proves the lemma. □

Thus a curve of constant width can be constructed as a smooth Legendrian curve $A(t)B(t) \subset M$, $t \in \mathbb{R}$, satisfying the monodromy condition $A(\pi) = B(0)$, $B(\pi) = A(0)$. Clearly, there is an abundance of such curves, in particular, analytic ones. This construction gives curves with cusps and inflections as well. (A similar approach is used in [2] to construct billiard tables that possess one-parameter families of periodic trajectories, the case of two-periodic trajectories being that of curves of constant width.)

To construct such a curve, take a closed wave front on the sphere with an odd number of cusps (say, an odd-cusped hypocycloid), place a geodesic segment of length $\pi/2$ orthogonally to the front, so that its midpoint is on the front, and use the front as a guide to move the geodesic segment all the way around until its end points swap their positions.

Curves of constant width $\pi/2$ in S^2 project to $\mathbb{RP}^2$ as contractible curves. Similarly one can construct self-dual noncontractible curves in $\mathbb{RP}^2$. For this, one needs to modify the

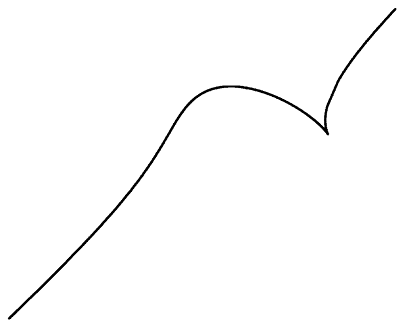

Fig. 8 Self-dual cubic curve with one inflection and one cusp

above monodromy condition: $A(\pi) = B(0)$, $B(\pi) = -A(0)$. A noncontractible curve necessarily has an odd number of inflections, and therefore, by self-similarity, an odd number of cusps. An example is a cubic curve $y = x^3$; after a projective transformation, this curve looks like shown in Fig. 8.

Rotationally symmetric curves Let us consider analogs of the polygons studied in Sect. 4. Assume that γ is a self-dual curve that lies in the affine plane (i.e., is disjoint from the line at infinity) and is star-shaped with respect to the origin O (i.e., the tangent lines to γ do not pass through O). We assume that γ is not a multiple of another curve. We allow γ to have inflections and cusps. Assume that the bilinear form F is not symmetric, so, by Proposition 16, $\varphi^2 \neq \mathrm{id}$.

Arguing as in Sect. 4, we choose a coordinate system in which the mapping G is a rotation through some angle α (the cases of Jordan blocks and complex angles in Lemma 11 are excluded because in these cases orbits of G would have accumulation points at infinity, in contradiction with Lemma 15). If α is π-irrational then the orbits of G are dense in a circle and, by Lemma 15, γ is a circle. Otherwise, $\alpha = 2\pi p/q$ where p and q are relatively prime. Thus, q is the least period of G, and hence of the circle diffeomorphism φ^2 as well. Choose a parameterization $\gamma(t)$ so that $\varphi^2(t) = t + 2\pi r/q$ where r and q are also relatively prime. To summarize, we have the following analog of Lemma 10.

Lemma 19 *One has* $G(\gamma(t)) = \gamma(t + \frac{2\pi r}{q})$.

Note that the rotation number of γ about the origin O equals the least positive k such that $kr \equiv p \bmod q$.

Let us describe an explicit construction of such self-dual curves. Let H be the rotation about the origin through angle $\pi p/q$ (so that $H^2 = G$) and set $c = \pi r/q$. As in Sect. 4, the projective duality has a simple geometric description in terms of the Euclidean metric: for a point A, the dual line A^* passes through the point $H(A)/|A|^2$ and is orthogonal to the vector $H(A)$.

Self-dual curves again can be described as Legendrian curves. Let M consist of pairs of vectors (u, v) such that $H(u) \cdot v = 1$. Define a contact structure on M by the condition $H(u) \cdot v' = 0$ (or, equivalently, $H(u') \cdot v = 0$); we leave it to the reader to check that this is indeed a contact structure. Let $(u(t), v(t))$ be a Legendrian curve in M satisfying the monodromy condition $u(t + c) = v(t)$, $v(t + c) = G(u(t))$. Then we can set: $\gamma(t) = u(t)$. The condition $H(u) \cdot v = 1$ implies that v belongs to the line u^*, and the Legendrian condition $H(u) \cdot v' = 0$ that this line is tangent to the curve γ at point $\gamma(t + c)$. Thus, γ is self-dual.

Now we give explicit formulas.

Proposition 20 *Let $\beta(t)$ be a smooth function such that $|\beta(t)| < \pi/4$ and $\beta(t+c) = -\beta(t)$. Let $\rho_1(t)$ and $\rho_2(t)$ satisfy the differential equations*

$$\rho_1' = (\beta' + 1)\tan 2\beta, \quad \rho_2' = (\beta' - 1)\tan 2\beta. \tag{1}$$

Then the curve $\gamma(t)$ whose polar coordinates are

$$\left(t - \beta(t) - \frac{\pi p}{q},\ e^{\rho_1(t)}\right)$$

is self-dual.

Proof Using the above notation, the polar coordinates of the points $H(u(t))$ and $v(t)$ are $(t-\beta,\ e^{\rho_1})$ and $(t+\beta,\ e^{\rho_2})$. The differential equations (1) are the Legendrian conditions $H(u)\cdot v' = 0$ and $H(u')\cdot v = 0$ which together imply that $H(u)\cdot v$ is constant. One needs to satisfy the monodromy conditions $\rho_1(t+c) = \rho_2(t)$, $\rho_2(t+c) = \rho_1(t)$. Due to (1), these equalities hold once one has

$$0 = \int_0^c \rho_1'(t)\,dt + \int_0^c \rho_2'(t)\,dt = 2\int_0^c \tan 2\beta\,d\beta = -\int_0^c d(\ln\cos 2\beta).$$

The latter is zero because $\beta(c) = -\beta(0)$, and we are done. □

Radon curves Let U be a Minkowski plane (two dimensional normed space) and let γ be its unit circle, a closed smooth strictly convex centrally symmetric curve centered at the origin. For a vector $u \in \gamma$, one defines its orthogonal complement as the tangent line to γ at u. This orthogonality relation is not symmetric, in general. A Minkowski plane is called a Radon plane, and the curve γ a *Radon curve*, if the orthogonality relation is symmetric. A Radon curve admits a one-parameter family of circumscribed parallelograms whose sides are orthogonal to each other. Introduced by J. Radon about 90 years ago, Radon curves abound (they have functional parameters), see [8] for a survey.

The relevance of Radon curves to our subject is the following statement.

Proposition 21 *Radon curves are projectively self-dual.*

Proof Let $[\cdot,\cdot]$ be an area element (linear symplectic structure) in U. Identify U^* with U using this area form: $u^* = [\cdot, u]$. With this identification, $u^\perp$ is a line ℓ, parallel to u, and such that $[v, u] = 1$ for every $v \in \ell$.

Similarly to the preceding discussion, a Radon curve can be realized as a curve $(u(t), v(t))$ in the space of pairs of nonzero vectors (u, v), tangent to the distribution given by the conditions $[u, v'] = 0 = [u', v]$ (these conditions mean that u is orthogonal to v and v is orthogonal to u), and satisfying the monodromy conditions: $u(t+\pi/2) = v(t)$, $v(t+\pi/2) = -u(t)$. The equalities $[u, v'] = 0 = [u', v]$ imply that $[u, v]$ is constant or, after rescaling γ, that $[u, v] = 1$. Therefore, $u^\perp$ is the tangent line to γ at point v, that is, γ is self-dual. □

Let us conclude with two remarks. First, Radon planes can be also characterized as the Minkowski planes for which the unit circle is the solution to the isoperimetric problem

(Busemann's theorem). Secondly, the outer billiard around a Radon curve possesses a one-parameter family of 4-periodic trajectories, see [4] for a study of such outer billiards in the context of sub-Riemannian geometry.

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