Around four vertices

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The classical theorem of Mukhopadhyaya and Kneser [1], [2] asserts that a plane closed non-self-intersecting curve has no fewer than four vertices, that is, points of extremal curvature. Recently, this theorem has received renewed attention in connection with investigations in projective and symplectic topology [3]. In this note we consider a problem generalizing the four-vertex theorem, give two proofs of the theorem, and establish an assertion stating that a function on a circle not having small harmonics must have many zeros.

1. We consider a closed convex plane curve γ, and denote by L the set of points in the plane from which the two tangents to the curve have the same length. If γ is a curve in general position, L consists of non-intersecting embedded curves. The boundary points of L lie on γ and are vertices of it, and the ends of L, that is, the infinitely distant points, correspond to the oriented diameters of γ (chords perpendicular to γ).

As in the vertex problem, the problem of equal tangents is projective-invariant: if x, y are points on γ that are points of contact of equal tangents to γ, and f is a Möbius transformation, then $f(x)$ and $f(y)$ are points of contact of equal tangents to the curve $f(\gamma)$. This follows from the existence of a circle tangent to γ at x and y.

2. Let A be a point outside γ, and let u and v be vectors along the right and left tangents from A to γ. We define two maps from the complement of the interior of γ to the plane: $\varphi(A) = u$, $\psi(A) = v$. A calculation establishes the following result.

***Proposition* 1.** (i) *φ and ψ preserve area.*

(ii) *Let $B(r)$ be the circle of radius r and centre the origin. The sets $\varphi^{-1}(B(r))$ and $\psi^{-1}(B(r))$ have the same centre of gravity.*

To prove the four-vertex theorem, we need the following result.

***Lemma* 2.** *The boundaries of two convex domains of the same area and with coincident centres of gravity intersect in no fewer than four points.*

If there are two points in all, the moments of inertia of the domains about the lines joining the points are not the same. We denote by $U(r)$ and $V(r)$ the domains $\varphi^{-1}(B(r))$ and $\psi^{-1}(B(r))$ complementing the interior of γ. The points of intersection of the boundaries of $U(r)$ and $V(r)$ lie in L, and to prove the four-vertex theorem it is enough to show that the number of such points is at least four for small r. This follows from the proposition and the lemma, since $U(r)$ and $V(r)$ are convex for small r.

It also follows from the proposition that there are two pairs of equal tangents to γ of length r, provided that r is small enough.

3. We have the following conjecture about the structure of L:

Conjecture. *L has four branches starting on γ and going off to infinity.*

We consider the torus that is the Cartesian product of γ with a natural parameter, and denote by Δ the set of pairs of points of contact of equal tangents to γ. Δ is an embedded curve that is symmetric with respect to the diagonal; therefore, after removal of contractible components, it consists of curves homologous either to the diagonal or the antidiagonal. The conjecture asserts that the latter possibility happens whenever the number of curves is at least two. The number of intersections of Δ with the curves parallel to the generators of the torus, and with the curves parallel to the diagonal, can be estimated.

***Proposition* 3.** *All of these numbers are at least two.*

For the proof, we observe that the desired points of intersection are critical points of two functions on circles: for the intersections of Δ with the generators of the torus, the function is the radius of the circle tangent to γ at a fixed point and passing through a point moving on γ; for the intersection of Δ with a curve parallel to the diagonal of the torus, it is the length of the chord subtending an arc of γ of fixed length.

If the conjecture about L is true, the second of the numbers mentioned is at least four.

4. Blaschke [4] deduced the four-vertex theorem from the fact that a certain function on a circle orthogonal to sine and cosine has no fewer than four critical points (the function is the radius of curvature, and the argument is the direction of the tangent to the curve; or the function could be the curvature $f+f''$, expressed in terms of the support function f of the curve). This can be generalized as follows:

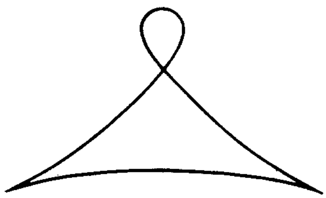
***Theorem* 4.** *Let $f(\alpha)$ be a function on a circle such that*

$$\int f(\alpha)\cos k\alpha\, d\alpha = \int f(\alpha)\sin k\alpha\, d\alpha = 0 \quad \textit{for} \quad k = 1, 2, \ldots, n.$$

Then $f(\alpha)$ has at least $2(n+1)$ critical points.

Proof. We regard f as the real part of the restriction to the boundary of a holomorphic function F defined in the unit disc. Then, ind grad Re $F \leqslant -n$ at the origin, and ind grad Re $F \leqslant 0$ at other critical points. Therefore, the number of revolutions of grad Re F on the boundary of the circle is no more than $-n$, so that grad Re F is perpendicular to the bounding circle at least $2(n+1)$ times.

As Arnol'd has observed, this result supplements Courant's theorem [5] (for the case of a circle) stating that the set of zeros of a function on a manifold that is a linear combination of the first n eigenfunctions of the Laplace operator divides the manifold into no more than n pieces. In this spirit, it is interesting to ask whether Theorem 4 can be extended to arbitrary manifolds?



5. In conclusion, we provide yet another proof of the four-vertex theorem. The vertices of a curve correspond to the cusps of its evolute. The evolute does not have inflections, and its notation number is equal to one. If there are two cusps, the evolute consists of a embedded arc joining the cusps and an arc not intersecting it with a point of self-intersection (see the figure). The length of the evolute is zero (the sign of the length of the arc changes as we pass through the cusps). However, for the curve in the figure, the convex non-self-intersecting arc is shorter than the second arc. It follows that there are no fewer than four cusps.

References

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