Homework problems in the Convexity course

1. Prove that $Conv(A)$ is convex for any set A.

2. Let $v_1, \ldots, v_k \in \mathbf{R}^n,\ c_1, \ldots, c_k \in \mathbf{R}$. Prove that the set

$$A = \{x \in \mathbf{R}^n \mid v_i \cdot x \le c_i \quad i = 1, \ldots, k\}$$

   is convex.

3. Show that the union of an open ball with any part of its boundary is convex. Is the same true for a cube?

4. Let V be the space of continuous functions $f : [0,1] \to \mathbf{R}$. Let

$$A = \{f \in V \mid |f(x)| \le 1 \quad \text{for all } x \in [0,1]\}$$

$$B = \{f \in V \mid f(x) \le 0 \quad \text{for all } x \in [0,1]\}.$$

   Are these sets convex?

5. Prove that the intersection of any number of convex sets is convex.

6. Let $T : V \to U$ be a linear map of vector spaces, $A \subset V$ a convex set. Show that $T(A)$ is convex.

7. If $A \subset B$ then $Conv(A) \subset Conv(B)$.

8. Show that $Conv(A) \cup Conv(B) \subset Conv(A \cup B)$.
   Does the equality hold?

9. Show that the Radon theorem is sharp for all dimensions.

10. Show that Helly's theorem fails for non-convex sets.

11. Construct infinitely many convex sets in $\mathbf{R}^n$ such that every $n+1$ intersect but the intersection of them all is empty.

12. In the theorem
Given a finite collection of convex sets $A_j, C \subset \mathbf{R}^n$, assume that for every $n+1$ of the sets A_j there exists a vector $x \in \mathbf{R}^n$ such that $x + C$ intersects these $n+1$ sets; then there exists $y \in \mathbf{R}^n$ such that $y + C$ intersects all the sets A_j,
replace the word "intersects" by:
a). "contains";
b). "is contained in";
and prove the resulting theorems.

13. Is the claim of the theorem
Given a (not necessarily uniform!) convex lamina in $\mathbf{R}^n$ of total mass 1, there exists a point y such that every half-space through y has the mass at least $1/(n+1)$,
sharp?

14. If a convex set in $\mathbf{R}^n$ is contained in the union of a finite collection of half-spaces then it is contained in the union of some $n+1$ of these half-spaces.

15. How does the support function $p(\phi)$ change when the origin is translated through vector (a, b)?

16. Compute the distance $h(\phi)$ between the tangency point of support line orthogonal to the direction ϕ and the base points of the perpendicular going from the origin to the support line.

17. Compute the area and the angles at the vertices of the Reuleaux triangle.

18. Construct a figure of constant width based on a regular n-gon for odd $n \geq 5$ (generalizing the Reuleaux triangle).

19. Let γ be a closed curve of constant width d. Show that the distance between any two points of γ is not greater than d.

20. Consider the lens-shaped domain bounded by two arcs of the circles centered at the opposite vertices of a square whose radii are equal to

the side of this square. Show that the perimeters of all rectangles circumscribed about this domain are the same.

21. Prove that if the perimeter of every rectangle circumscribed about a smooth closed convex curve with positive curvature equals L then the perimeter length of this curve is $\pi L/4$.

22. Deduce the area formula for a closed convex smooth curve in terms of the support function using the Green's theorem $A = (1/2)\int_0^2 \pi xdy - ydx$.

23. Describe the equality case in Wirtinger's inequality.

24. Prove that function represented by the Fourier series is real valued if and only if $c_k = \bar{c}_{-k}$.

25. Assuming the isoperimetric inequality or all plane domains, solve the Dido problem.

26. Find a linear harmonic $c + a\cos\phi + \sin\phi$ (that is, find a, b, c) that has roots at points $\alpha, \beta \in [0, 2\pi)$.

27. Prove that a trigonometric polynomial of degree 1 has at most 2 roots on the period interval $[0, 2\pi)$.

28. Let $f(\phi) = a\cos 3\phi + b\sin 3\phi + ...$ be a trigonometric polynomial whose Fourier expansion starts with the third harmonics. Show that f has at least 6 roots on the period interval $[0, 2\pi)$.

29. Prove that critical points of the distance function $d(O, \gamma(\phi))$ are in 1-1 correspondence with the normals to γ from the origin.

30. Consider a plane domain bounded by a smooth closed curve with positive curvature γ. Show that there exists a point y in the plane such that there are at least four normals from y to γ.

31. Find and sketch the evolute of a parabola.

32. Prove that the number of cusps of the evolute of a closed curve is always even.

33. Investigate the involute of a cubic parabola $y = x^3$: sketch it and, if possible, describe by formulas.

34. Denote by $f_a(x)$ the quadratic Taylor polynomial of the function $f(x) = x^3$ at point $a \in \mathbf{R}$. Prove that the graphs of the functions $f_a(x)$ and $f_b(x)$ are disjoint for all $a \neq b$.

35. A curve is given by its support function $p(\phi)$. How to detect a cusp of the curve in terms of $p(\phi)$?

36. Show that if A is convex then $Int(A)$ is also convex.

37. Let $\Delta \subset \mathbf{R}^n$ be an n-dimensional simplex. Show that $Int(\Delta)$ is not empty.

38. Describe the faces of 3-dimensional cube.

39. Describe the faces of an n-dimensional simplex.

40. Let F_1 and F_2 be faces of a closed convex set. Prove that $F_1 \cap F_2$ is also a face.

41. A face of a closed convex set is closed and convex.

42. Let B be a face of A and C be a face of B. Is C necessarily a face of A?

43. Prove that every compact convex set has an exposed point.

44. Show that an exposed point is an extreme point.

45. Consider a map on a sphere (a planar graph). Prove that if each face has order of at least 3 than the average order of a vertex is at most 6.

46. Prove that every polyhedron in 3-space has two faces with the same number of sides.

47. Consider a convex polytope in $\mathbf{R}^3$. Prove that the average order of a vertex and the average order of a face are at most 6. Give an example of a convex polytope such that the average order of a face is greater than 5.5.

48. Prove that every convex polyhedron in 3-space has either a triangular face or a vertex that is incident to three edges (or both).

49. Prove that the Minkowski sum of convex bodies is convex.

50. If A is convex body, $\lambda, \mu > 0$ then $(\lambda + \mu)A = \lambda A + \mu A$. Is this still true if λ or μ are negative?

51. Let γ_1 and γ_2 be convex plane curves, and P_1, P_2 be convex n-gons, circumscribed about these curves and having parallel corresponding sides. Prove that $P_1 + P_2$ is circumscribed about $\gamma_1 + \gamma_2$.

52. Let A be a triangle in $\mathbf{R}^3$ and B be a square in parallel plane. Draw the polyhedron $(1-t)A + tB$ where $t \in [0, 1]$.

53. Prove that $Vol((1-t)A + tB) \geq Vol(A)^{1-t} Vol(B)^t$.

54. Let X be a convex polygon in the plane or a convex polyhedron in 3-space. Let $f(X) = \{2x - y \mid x, y \in X\}$ be the union of reflections of X in all its points.
a). Prove that $f(X)$ is a convex polygon/polyhedron.
b). Let X be a square. Describe the sequence $f(X), f^2(X), f^3(X), \ldots$. Same for a cube.
c). Same question as in b) with a triangle and a tetrahedron.

55. Find the mixed area of an equilateral triangle and its centrally symmetric image.

56. Prove the Minkowski theorem in dimension two.

57. Analyze the area of the polygon bounded by the lines

$$y = px, \ y = qx, \ x = a, \ y = b, \ x + y = c$$

as a function of the distances from these lines to the origin.

58. Prove Euler's formula: if $V(z_1, \ldots, z_p)$ is a homogeneous function of degree n then

$$\sum_1^p z_i \frac{\partial V}{\partial z_i} = nV.$$

59. Prove that a simplicial and simple polyhedron is either a simplex or a polygon.

60. Let (f_0, f_1, f_2) be the f-vector of a convex polyhedron in 3-space. Prove:

$$4 \leq f_0 \leq \frac{2}{3} f_1, \ \ 4 \leq f_2 \leq \frac{2}{3} f_1.$$

61. Compute the h-vector for a simplex.

62. Show that the relation $h_0 = h_n$ is equivalent to the Euler formula for the respective polyhedron.

63. Given a simple n-dimensional polyhedron, prove that $h_k > 0$ for each $k = 0, \ldots, n$.

64. Let P be a convex polyhedron in $\mathbf{R}^3$. Can every plane section of P (not through a vertex) be:
(i) a triangle?
(ii) a quadrilateral?
(iii) an odd-sided polygon?

65. Prove that any $k \leq n$ distinct points on the Veronese curve in $\mathbf{R}^n$ are affinely independent.