Around Helly's Theorem

1. (Jung's Theorem). Let A be a convex compact set in $\mathbf{R}^n$ whose diameter (maximal distance between its points) is 2. Prove that A is contained in a ball of radius $\sqrt{(2n)/(n+1)}$, and this is a sharp estimate.

2. (Krasnoselsky's Theorem). An art gallery X is a (non-convex!) plain domain (polygonal, if you wish). For every three points a, b, c of X there exists a point x in X such that the segments xa, xb, xc are contained in X. Show that X is star-shaped: there is a point $x \in X$ such that for each $y \in X$, the segment xy is contained in X.

Properties of plane convex figures

3. Let A be a convex plane domain bounded by a smooth curve. Let the width of A be equal to 1. Prove that the distance from the center of mass of A to every tangent line to its boundary is not less than 1/3.

4. Let A be a bounded convex plane domain of area 1, and L be a line through the center of mass of A. Prove that the area cut by L from A is between 4/9 and 5/9, and these bounds are sharp.

5. (Kovner's Theorem). Let A be a bounded convex plane domain of area 1. Prove that A contains a centrally-symmetric figure of area at least 2/3.

6. Let A be a bounded convex plane domain of diameter 1. Prove that A is contained inside a regular hexagon whose sides are equal to $1/\sqrt{3}$.

Figures of constant width

7. Let A be a plane domain of constant width w. Prove that the inscribed and circumscribed circles of A are concentric, and the sum of their radii equals w.

8. Prove that the Reuleaux triangle has the smallest area among curves of a given constant width.

Isoperimetric inequality

9. Let P be a convex polygon, inscribed into a circle. Show that P has a greater area than any other polygon with the same sides as P (you may use the isoperimetric inequality).

10. Prove that the regular polygon has the greatest area among all convex n-gons with a given perimeter.

Support function

11. Find the support function of an ellipse.

12. Let $f(x)$ be a smooth function, let n be an even number, and let $g_a(x)$ be the Taylor polynomial of degree n for $f(x)$ at point a. Assume that I is an interval such that $f^{(n+1)}(x) \neq 0$ or every $x \in I$. Prove that for every distinct $a, b \in I$, the graphs of the polynomials $g_a(x)$ and $g_b(x)$ are disjoint.

13. Prove that a trigonometric polynomial of degree n has at most $2n$ roots.

Polynomials and convexity

14. (Gauss-Lucas) The roots of the derivative of a complex polynomial belong to the convex hull of the roots of the polynomial.

Around Cauchy theorem and the 4-vertex theorem

15. Given two plane ovals, let ds and ds_1 be the arc length elements at points with parallel and equally oriented exterior normals. Then the ratio ds_1/ds has at least four extrema.

16. Let P and P' be plane convex n-gons, $n \geq 4$, whose sides have lengths $\ell_1, \ldots, \ell_n$ and $\ell'_1, \ldots, \ell'_n$. Assume that the corresponding sides of the polygons are parallel to each other. Consider the cyclic sequence

$$a_i = \left(\frac{\ell'_{i+1}}{\ell_{i+1}} - \frac{\ell'_i}{\ell_i}\right)\left(\frac{\ell'_{i-1}}{\ell_{i-1}} - \frac{\ell'_i}{\ell_i}\right).$$

Prove that either $a_i = 0$ for all i or $a_i > 0$ for at least four values of i.

17. Prove a continuous analog of the Arm Lemma: given two smooth convex arcs $\gamma_1(s)$ and $\gamma_2(s)$ of equal lengths and parameterized by arc length, if their curvatures satisfy the inequality $k_1(s) \geq k_2(s)$ for all s then the chord subtended by γ_2 is not less than that subtended by γ_1.

Minkowski sum

18. Show that every convex polygon is the Minkowski sum of triangles and segments.

19. Show that every convex quadrilateral has a unique representation as the Minkowski sum of triangles and segments, but there exist pentagons that have different representations as the Minkowski sums of triangles.

20. Let $X, Y \subset \mathbf{R}^n$ be compact convex bodies. Define the Hausdorff distance $d(X, Y)$ as the smallest r such that $X \subset Y + rB$ and $Y \subset X + rB$ where B is the unit ball. Prove that d satisfies the triangle inequality.

21. Let $X, Y \subset \mathbf{R}^2$ be convex domains bounded by smooth curves. Prove that if $d(X, Y) \leq r$ then the difference of the perimeter lengths of the curves is not greater than $2\pi r$, and the difference of the areas of X and Y is not greater than $Lr + \pi r^2$ where L the greater of these perimeter lengths.

Around Minkowski Theorem

22. Let $P \subset \mathbf{R}^n$ be a convex polyhedron such that for every $n-1$-dimensional face of P there exists a parallel face with the same area. Prove that P is centrally symmetric.

23. Suppose that each $n-1$-dimensional face of an n-dimensional convex polyhedron is centrally symmetric. Prove that P is also centrally symmetric.

24. Let $P \subset \mathbf{R}^n$ be a convex polyhedron such that its parallel translations tile the space. Prove that the opposite $n-1$-dimensional faces of P have equal areas.