Math 527, Fall 2008

Second Partial Exam

Solve as many problems as you can.

Due Friday December 19, 2008 at 5 PM

Question 1. (a) Construct a 2-dimensional CW complex X, whose **fundamental group** is $\pi_1(X) = \langle a, b, c, d, e, f | a^2bc^3b^{-1}, e^2f^2e^2, d^5 \rangle$.

(b) Compute $H_1(X)$.

Question 2. Let M_g be the orientable closed surface of genus g and N_h be the nonorientable closed surface of genus h.

(a) Can M_3 be a **covering space** of M_2 ?

(b) Can M_g be covered by N_h for any pair (g, h) ?

(c) Can M_2 cover N_4 ?

Question 3. The group $SU(2)$ is a topological subspace of $\mathbb{C}^4$. Compute its homology groups.

Question 4. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their **universal covering spaces** do not.

Question 5. Compute the **Euler characteristic** of the 2-sphere S^2, the torus $S^1 \times S^1$ and the orientable surface M_g of genus g.

Question 6. A simple graph is an undirected graph that has no loops and no more than one edge between any two different vertices. A complete graph is a simple graph in which every pair of distinct vertices is connected by an edge.

(a) Suppose we build S^2 from a finite collection of polygons by identifying edges in pairs. This way, we get a CW structure on S^2. Show that there is a way to do that so that the 1-skeleton is the complete graph over four vertices.

(b) On the other hand, show that there is no way the 1-skeleton can be the complete graph over five vertices.

Question 7. For SX the suspension of X, show by a **Mayer-Vietoris sequence** that there are isomorphisms $\tilde{H}_n(SX) \cong \tilde{H}_{n-1}(X)$ for all n.

Question 8. (a) Write down the statement of the Borsuk-Ulam theorem.

(b) Write down the statement of the Brouwer fixed point theorem for maps $f : D^n \to D^n$.

(c) Prove the Brouwer fixed point theorem by applying **degree theory** to the map $S^n \to S^n$ that sends both the northern and southern hemisphseres of S^n to the southern hemisphere via f

(d) Prove the Ham sandwich theorem : Given n compact sets $A_1, \ldots, A_n$ in $\mathbb{R}^n$, there always exists a hyperplane dividing each of them into two subsets of equal measure.

(e) What is know as the Teller-Ulam design ?