Math 527, Fall 2008
First Partial Exam

To get a perfect score, you must provide a complete answer for at least one question in each pair of questions numbered $2n-1, 2n$ (where $n \in \{1, \ldots, 5\}$).

Question 1. Write down a definition and an example of each of the following concepts :
(a) a directed set ;
(b) a cofinal map between two directed sets ;
(c) a separable topological space ;
(d) a completely regular space ;
(e) a partition of unity ;
(f) the axiom of choice.

Question 2. Write down explicitely the celebrated statements which are referred to as :
(a) Tychonoff's theorem ;
(b) Urysohn's lemma ;
(c) Urysohn's metrization theorem ;
(d) Tietze's extension theorem ;
(e) the Stone-Weierstraß theorem ;
(f) the Weierstraß approximation theorem.

Question 3. Let $\left((X_n, d_n)\right)_{n=1}^{\infty}$ be a sequence of metric spaces. Is the product topology on $X = \prod_{n=1}^{\infty} X_n$ metrizable ?

Question 4. Give an example of a path connected space which is not locally connected.

Question 5. Describe the one-point compactification of the complement of the equator in the sphere S^2.

Question 6. Can a Peano curve (i.e. a continuous map from $[0, 1]$ onto $[0, 1] \times [0, 1]$) be bijective ?

Question 7. Consider the product topology on the product $X \times Y$ of two topological spaces X and Y. Is it true that every closed subspace of $X \times Y$ is of the form $F \times G$, where F is a closed subspace of X and G is a closed subspace of Y ?

Question 8. Let A be a directed set, and let $f : \mathbb{N} \to A$ be a strictly increasing function (i.e. $f(i) < f(j)$ if $i < j$). Given a Cauchy net $\{x_\alpha\}_{\alpha \in A}$ in a metric space X, consider the sequence $\mathbb{N} \to X : n \mapsto x_{f(n)}$. Is it necessarily a Cauchy sequence ?

Question 9. Let X, Y and Z be three nonempty topological spaces.
(a) Could it happen that $X \times Y$ is homeomorphic to $X \times Z$ while Y is not homeomorphic to Z ?
(b) Same question with $X = [0, 1]$.
(c) Same question with $X = [0, 1]$ and $Y = \{*\}$, the one-point space.

Question 10. Let $(K, \mathcal{T})$ be a compact Hausdorff space, and let $\mathfrak{m} \subset C(K, \mathbb{F})$. Prove the equivalence of the following three assertions.
(a) There exists $x \in K$ such that $\mathfrak{m} = \{f \in C(K, \mathbb{F}) | f(x) = 0\}$.
(b) There is a nonzero, multiplicative linear map $\phi : C(K, \mathbb{F}) \to \mathbb{F}$ such that $\mathfrak{m} = \{f \in C(K, \mathbb{F}) | \phi(f) = 0\}$.
(c) The subset $\mathfrak{m}$ is a maximal ideal of $C(K, \mathbb{F})$.