

MATH 527 (Fall 2008)

Homework #7

Due Monday Dec 1.

Hatcher, pages 131-133: exercises 3, 4, 7, 9, 10, 11, 12, 15, 16, 18

Show that S^3 is made of two solid tori glued along their outer surface.

Homework #6

Due Monday Oct 13.

Hatcher, pages 18-20: exercises 6, 11, 16, 23, 28

Homework #5

Due Wednesday Oct 8.

Hatcher, pages 18-20: exercises 3, 5, 9, 17, 18

Homework #4

Due Monday Sep 29.

In Runde's book

page 40: exercise 3

page 51: exercises 2, 3, 7

pages 78-79: exercises 11, 12

page 99: exercises 3, 4, 5, 6, 7, 9

Homework #3

Due Monday Sep 22.

In Runde's book

pages 88-89: exercises 1, 3, 6, 7, 8, 9, 10

page 106: exercises 1, 2, 3, 4, 8

Homework #2

Solve ten of the following fourteen problems. Solutions due Monday Sep 15.

Exercise 1. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces, and let X_1 and X_2 be two subspaces of X such that $X_1 \cup X_2 = X$ which are both open or both closed. Show that $f : X \rightarrow Y$ is continuous if and only if $f|_{X_j}$ is continuous for $j = 1, 2$.

Exercise 2. Let (X, \mathcal{T}) be a first countable topological space. Show that, for any $S \subset X$, the closure \overline{S} of S consists of those points $x \in X$ such that there is a sequence in S converging to x .

Exercise 3. Let (X, \mathcal{T}) be a topological space, and let \approx be an equivalence relation on X . Show that the quotient topology on X/\approx is the finest topology on X/\approx making the quotient map

$$X \rightarrow X/\approx: x \mapsto [x]$$

continuous.

Exercise 4. Let (X, \mathcal{T}) be a topological space, and let $f, g : X \rightarrow \mathbb{R}$ be continuous. Show that $\max\{f, g\}$ and $\min\{f, g\}$ are continuous.

Exercise 5. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces such that Y is Hausdorff, let D be dense in X , and let $f, g : X \rightarrow Y$ be continuous functions such that $f|_D = g|_D$. Show that $f = g$. What happens if we drop the demand that Y be Hausdorff?

Exercise 6. Let R be a commutative ring with identity. Prove that $\text{Spec}(R)$ is compact by showing that it has the finite intersection property.

Exercise 7. Let $S \neq \emptyset$ be a set, and let (X, d) be a metric space. A function $f : S \rightarrow X$ is said to be bounded if

$$\sup_{x, y \in S} d(f(x), f(y)) < \infty.$$

We consider the metric

$$D(f, g) = \sup_{x \in S} d(f(x), g(x)) \quad \text{where } f, g \in B(S, X)$$

on the space $B(S, X)$ of all bounded functions from S to X .

Let Y be the subspace of $B(\mathbb{N}, \mathbb{C})$ consisting of those sequences tending to zero. Show that Y is separable.

Exercise 8. If (X, d) is a separable metric space and Y is a subspace of X , prove that Y is also separable.

Exercise 9. Let (X, d) be a metric space, and let $S \subset X$. The distance of $x \in X$ to S is defined as

$$\text{distance}(x, S) = \begin{cases} \inf_{y \in S} d(x, y) & \text{if } S \neq \emptyset \\ \infty & \text{if } S = \emptyset \end{cases}.$$

If $S \neq \emptyset$, show that the function

$$X \rightarrow \mathbb{R} : x \mapsto \text{distance}(x, S)$$

is continuous.

Exercise 10. Show that a discrete metric space (X, d) is compact if and only if X is finite.

Exercise 11. Let (X, d) be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in X with limit x_0 . Show that the subset $\{x_0, x_1, x_2, \dots\}$ of X is compact.

Exercise 12. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \rightarrow Y$ is called uniformly continuous if, for each $\epsilon > 0$, there is $\delta > 0$ such that $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \delta$. Show that any continuous function from a compact metric space into another metric space is uniformly continuous.

Exercise 13. Let (X, d) be a metric space. The diameter of a subset $S \neq \emptyset$ of X is defined as

$$\text{diameter}(S) = \sup_{x, y \in S} d(x, y).$$

Show that

$$\text{diameter}(S) = \inf\{r > 0 : S \subset B_r(x) \text{ for all } x \in S\}.$$

Exercise 14. *Lebesgue's covering lemma.* Let (K, d) be a compact metric space, and let \mathcal{U} be an open cover of K . Show that there is a number $L(\mathcal{U}) > 0$ (the Lebesgue number of \mathcal{U}) such that any $\emptyset \neq S \subset K$ with $\text{diameter}(S) < L(\mathcal{U})$ is contained in some $U \in \mathcal{U}$.

Homework #1

Due Wednesday Sep 10.

In Runde's book

page 34: exercises 3, 4

page 71: exercises 3, 4(b), 5, 8

page 78: exercises 3, 5, 6, 10