Math 140.043, Fall 2008
Key to Quiz #3

Question 1. True or false? Explain why.

(a) If f is continuous on $[-1,1]$ and $f(-1)=4$, $f(1)=3$, then there exists a number c in $(-1,1)$ such that $f(c)=\pi$.

TRUE Since the number π lies between $f(-1)$ (which is known to be 4) and $f(1)$ (which is known to be 3) and f is continuous on the interval $[-1,1]$, the intermediate value theorem implies that there exists a number c in the interval $(-1,1)$ such that $f(c)=\pi$.

(b) The equation $x^{10}-10x^2+5=0$ has a root in the interval $(0,2)$.

TRUE Setting $f(x)=x^{10}-10x^2+5$, we have $f(0)=5>0$, $f(1)=-4<0$ and $f(2)=629>0$. Since 0 lies between $f(0)$ (which is known to be 5) and $f(1)$ (which is known to be -4) and f is continuous on the interval $[0,1]$, the intermediate value theorem implies that there exists a number c in the interval $(0,1)$ such that $f(c)=\pi$. Note that, since 0 lies between -4 and 629, there is another root c' of the polynomial in the interval $(1,2)$.

(c) If $p(x)$ is polynomial, then $\lim_{x\to b} p(x)=p(b)$.

TRUE A polynomial is continuous everywhere.

(d) If the line $x=1$ is a vertical asymptote of $y=f(x)$, then f is <u>not</u> defined at 1.

FALSE The line $x=1$ is a vertical asymptote of the function

$$f(x)=\begin{cases}3 & \text{if } x\le 1\\ \frac{1}{x-1} & \text{if } x>1\end{cases}$$

which is defined at 1, where it takes the value $f(1)=3$.

(e) If f is continuous at 5 and $f(5)=2$ and $f(4)=3$, then $\lim_{x\to 2} f(4x^2-11)=2$.

TRUE Indeed we have

$$\lim_{x\to 2} f(4x^2-11)=f(\lim_{x\to 2} 4x^2-11)=f(4\lim_{x\to 2}(x^2)-11)=f(4(\lim_{x\to 2} x)^2-11)$$
$$=f(4(2)^2-11)=f(5)=2,$$

where we have used the continuity of f to permute f with the limit symbol, the limit laws, the continuity of the square function, the stupid limit $\lim_{x\to 2} x=2$ and the given fact $f(5)=2$.

(f) If $f(x)\ge 1$ for all x and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x)\ge 1$.

TRUE

(g) If $f(x) > 1$ for all x and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 1$.

FALSE   Consider the function

$$f(x) = \begin{cases} x^2+1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0. \end{cases}$$

It is easy to see that $f(x) > 1$ everywhere and that $\lim_{x\to 0} f(x) = 1$.

(h)

$$\lim_{x\to 1} \frac{x^2+3x-4}{x^2+4x-5} = \frac{\lim_{x\to 1} x^2+3x-4}{\lim_{x\to 1} x^2+4x-5}$$

FALSE   Indeed

$$\lim_{x\to 1} \frac{x^2+3x-4}{x^2+4x-5} = \lim_{x\to 1} \frac{(x-1)(x+4)}{(x-1)(x+5)} = \lim_{x\to 1} \frac{(x+4)}{(x+5)} = \frac{5}{6}$$

while

$$\frac{\lim_{x\to 1} x^2+3x-4}{\lim_{x\to 1} x^2+4x-5} = \frac{1^2+3\cdot 1-4}{1^2+4\cdot 1-5} = \frac{0}{0} \text{ does not exist!}$$

Question 2. If f is differentiable at a, which of the following statements is / are right?

(a) $f'(a) = \lim_{h\to a} \frac{f(a+h)-f(a)}{h}$ WRONG

(b) $f'(a) = \lim_{x\to 0} \frac{f(x)-f(a)}{x-a}$ WRONG

(c) $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ RIGHT

(d) $f'(a) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{h}$ WRONG

(e) $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ RIGHT

(f) $f'(a) = \lim_{h\to 0} \frac{f(h)-f(a)}{h}$ WRONG

Use the definition above to compute the derivative of the function $f(x) = \frac{1}{1+x}$ at the number a, where $a \neq -1$.

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\frac{1}{1+x}-\frac{1}{1+a}}{x-a} = \lim_{x\to a} \frac{\frac{(1+a)-(1+x)}{(1+x)(1+a)}}{x-a} = \lim_{x\to a} \frac{1+a-1-x}{(1+x)(1+a)(x-a)}$$

$$= \lim_{x\to a} \frac{-(x-a)}{(1+x)(1+a)(x-a)} = \lim_{x\to a} \frac{-1}{(1+x)(1+a)} = \frac{-1}{(1+a)(1+a)} = \frac{-1}{(1+a)^2}$$