

LECTURE 21

Even Better (Mechanical) Vibrations

We've finished discussing what happens when we have free vibrations. In this situation, the differential equation modeling the displacement of an object attached to a spring is

$$mu'' + \gamma u' + ku = 0,$$

where m is the mass of the object, γ is the damping coefficient of any damper that might be present, and k is the spring coefficient.

When there is no damping ($\gamma = 0$), our solution is

$$\begin{aligned} u(t) &= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \\ &= R \cos(\omega_0 t - \delta). \end{aligned}$$

In this case, the solution oscillates with amplitude R and frequency ω_0 , which is called the *natural frequency* of the system.

If we hook up a damper with damping coefficient γ , the solution depends on what γ is. If $\gamma = \gamma_{CR} = 2\sqrt{mk}$, we have a critically damped system, and, as the only root for the characteristic equation is $r = -\frac{\gamma}{2m}$, the solution is

$$u(t) = c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t}.$$

If $\gamma > \gamma_{CR}$, we have an overdamped system, and the characteristic equation has real and distinct roots $r_1 \neq r_2$, where both roots are negative. This gives a solution of

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

In both the critically damped and overdamped cases, the damping is strong enough to cancel out almost all oscillation (there may be a single crossing of the equilibrium position, but beyond that, the entire system settles back to equilibrium). If $\gamma < \gamma_{CR}$, though, we have an underdamped situation, where the characteristic equation has complex roots $r_{1,2} = \alpha \pm i\beta$, where $\alpha < 0$. This means the solution has the form

$$\begin{aligned} u(t) &= c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t) \\ &= R e^{\alpha t} \cos(\beta t - \delta). \end{aligned}$$

Here the solution still goes to the equilibrium $u = 0$ in the limit, but it does so after oscillating (this is the only damped case where we actually get true "vibrations.") We can't call the solution periodic, however, since the amplitude is changing. So we call it quasi-periodic, with quasi-frequency β .

1. Forced, Undamped Motion

It's now time to discuss what happens if we allow some external force $F(t)$ to act on our system. This function $F(t)$ is called, appropriately enough, the *forcing function*. We start by considering the undamped case. Our differential equation is

$$mu'' + ku = F(t).$$

This is a nonhomogeneous equation, which means that the general solution has the form

$$u(t) = u_c(t) + U_p(t),$$

where $u_c(t)$ is the general solution to the associated homogeneous equation, which is just the analogous free, undamped case, and $U_p(t)$ is a particular solution to the nonhomogeneous equation, which we can find using undetermined coefficients (if $F(t)$ is of an appropriate form) or variation of parameters.

We will restrict our attention to the interesting case where

$$F(t) = F_0 \cos(\omega t) \quad \text{or} \quad F(t) = F_0 \sin(\omega t).$$

In other words, the force that we're applying to our spring-mass system is a simple periodic function with frequency ω . In the subsequent discussion, I'll assume that $F(t) = F_0 \cos(\omega t)$, but everything is completely analogous if $F(t)$ is a sine function. So, the equation that we'll consider is just

$$mu'' + ku' = F_0 \cos(\omega t).$$

Since the complimentary solution is the solution to the analogous free, undamped equation, as noted earlier it is just

$$u_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t),$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency.

We can use the method of undetermined coefficients for this nonhomogeneous term $F(t)$. The initial guess for the form of the particular solution is

$$U_p(t) = A \cos(\omega t) + B \sin(\omega t).$$

We need to be careful, however! This guess is fine if $\omega \neq \omega_0$, but if the frequency of the forcing function is the same as the natural frequency, this guess is precisely the complimentary solution u_c . Thus, if $\omega = \omega_0$, we'll need to add a factor of a t to our guess.

As a result, we have two cases we need to consider.

(1) $\omega \neq \omega_0$

In this case, our initial guess is distinct from the complimentary solution, and so the form of our particular solution will be

$$U_p(t) = A \cos(\omega t) + B \sin(\omega t).$$

Differentiating this guess and plugging it in to our differential equation, we get

$$\begin{aligned} m\omega^2(-A \cos(\omega t) - B \sin(\omega t)) + k(A \cos(\omega t) + B \sin(\omega t)) &= F_0 \cos(\omega t) \\ (-m\omega^2 A + kA) \cos(\omega t) + (-m\omega^2 B + kB) \sin(\omega t) &= F_0 \cos(\omega t). \end{aligned}$$

Setting coefficients equal, we get

$$\begin{aligned} (\cos(\omega t)) : \quad & (-m\omega^2 + k)A = F_0 & \Rightarrow & \quad A = \frac{F_0}{k - m\omega^2} \\ (\sin(\omega t)) : \quad & (-m\omega^2 + k)B = 0 & \Rightarrow & \quad B = 0. \end{aligned}$$

So our particular solution is

$$\begin{aligned} U_p(t) &= \frac{F_0}{k - m\omega^2} \cos(\omega t) \\ &= \frac{F_0}{m \left(\frac{k}{m} - \omega^2 \right)} \cos(\omega t) \\ &= \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t). \end{aligned}$$

Notice that the amplitude of the particular solution is dependent on two things: the amplitude of the forcing function F_0 and the difference between the natural frequency and the forcing frequency.

We can write our displacement function in two forms, depending on which form we use for the complimentary solution.

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

$$u(t) = R \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

Again, we'd get an analogous solution if our forcing function had been $F(t) = F_0 \sin(\omega t)$.

The key feature of this case can be seen from the second form of our displacement function; namely, we have two cosine functions with different frequencies. These will interfere with each other, causing the net oscillation to vary between bouts of greater and lesser amplitude. This phenomenon is called "beats," in analogy to the musical phenomenon. Think of this case like hitting a tuning fork when another one is already ringing; the increasing/decreasing volume that one hears in that situation as the waves interfere with each other is the same as what happens to the size of the oscillations for this spring-mass system.

(2) $\omega = \omega_0$

If the frequency of the forcing function is the same as the natural frequency, our guess for the particular solution has to become

$$U_p(t) = At \cos(\omega_0 t) + Bt \sin(\omega_0 t).$$

Differentiating, plugging in, and simplifying gives

$$(-m\omega_0^2 + k)At \cos(\omega_0 t) + (-m\omega_0^2 + k)Bt \sin(\omega_0 t) + 2m\omega_0 B \cos(\omega_0 t) - 2m\omega_0 A \sin(\omega_0 t) = F_0 \cos(\omega_0 t).$$

To begin further simplification, let's recall that $\omega_0^2 = \frac{k}{m}$, so $m\omega_0^2 = k$. This means the first two terms will actually vanish (as we would expect, since there are no analogous terms on the right hand side), and we get

$$2m\omega_0 B \cos(\omega_0 t) - 2m\omega_0 A \sin(\omega_0 t) = F_0 \cos(\omega_0 t).$$

Now, let's set coefficients equal.

$$\begin{array}{lll} (\cos(\omega_0 t)) : & 2m\omega_0 B = F_0 & B = \frac{F_0}{2m\omega_0} \\ (\sin(\omega_0 t)) : & -2m\omega_0 A = 0 & A = 0 \end{array}$$

Thus the particular solution in this case is

$$U_p(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$$

and the displacement is

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t)$$

or

$$u(t) = R \cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$

What stands out the most about this equation? Notice that as $t \rightarrow \infty$, $u(t) \rightarrow \infty$ due to the form of the particular solution. Thus, in the case where the forcing frequency

is the same as the natural frequency, the oscillation will have an amplitude that continues to increase for all time since the external force adds energy to the system in a way that reinforces the natural motion of the system.

This phenomenon is called *resonance*. Resonance is the phenomenon behind microwave ovens: the microwave radiation strikes the water molecules in what's being heated at their natural frequency, causing them to vibrate faster and faster, which generates heat. A similar phenomenon occurs in the Bay of Fundy, where tidal forces cause the ocean to resonate, yielding larger and larger tides. Resonance in the ear causes us to be able to distinguish between different tones in sound.

A common example that's cited of resonance is the collapse of the Old Tacoma Narrows Bridge. This is incorrect, however: the oscillation that led to the collapse of the bridge was from a far more complicated phenomenon than the simple resonance we're considering now.

In general, for engineering purposes, resonance is something that we'd like to avoid unless we thoroughly understand the situation and what the effect of the resonance will be.

There's no reason to suppose that the forcing function was sinusoidal as we did. It provides us with enough interesting behavior to be worthwhile, though. If you're faced with a forcing function that is of a different sort, you can still approach it using these techniques, though the phenomena that result will be different.

So, to sum up, when we drive our system at a different frequency than the natural frequency, the two frequencies interfere and we observe beats in the motion. When the system is driven at its natural frequency, the natural motion of the system is reinforced, causing the amplitude of the motion to increase to infinity.

Let's do an example.

EXAMPLE 21.1. *A 3 kg object is attached to a spring, which it stretches by 40 cm. There is no damping, but the system is forced with forcing function*

$$F(t) = 10 \cos(\omega t)$$

such that the system will experience resonance. If the object is initially displaced 20 cm downward and given an initial upward velocity of 10 cm/sec, find the displacement at any time t .

We need to be aware of units: we should convert all lengths to meters. The first thing to do is to find k .

$$k = \frac{mg}{L} = \frac{(3)(10)}{.4} = 75$$

Next, we're told that the system experiences resonance. Thus, the forcing frequency ω must be the same as the natural frequency ω_0 .

$$\omega = \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{75}{3}} = 5$$

Thus our initial value problem is

$$3u'' + 75u = 10 \cos(5t) \quad u(0) = .2 \quad u'(0) = -.1.$$

The complimentary solution is the general solution to the associated free, undamped case. As we've computed the natural frequency already, the complimentary solution will just be

$$u_c(t) = c_1 \cos(5t) + c_2 \sin(5t).$$

The particular solution (using the formula derived above) is

$$U_p(t) = \frac{1}{3}t \sin(5t),$$

and so the general solution is

$$u(t) = c_1 \cos(5t) + c_2 \sin(5t) + \frac{1}{3}t \sin(5t).$$

The initial conditions give that $c_1 = \frac{1}{5}$ and $c_2 = -\frac{1}{50}$, so the displacement can be given as

$$u(t) = \frac{1}{5} \cos(5t) - \frac{1}{50} \sin(5t) + \frac{1}{3}t \sin(5t).$$

Let's convert the first two terms to a single cosine.

$$R = \sqrt{\left(\frac{1}{5}\right)^2 + \left(-\frac{1}{50}\right)^2} = \sqrt{\frac{101}{2500}}$$

$$\tan(\delta) = \frac{-\frac{1}{50}}{\frac{1}{5}} = -\frac{1}{10}$$

Looking at the signs of c_1 and c_2 , we see that $\cos(\delta) > 0$ and $\sin(\delta) < 0$. Thus δ is in Quadrant IV, and so we can just take the arctangent.

$$\delta = \arctan\left(-\frac{1}{10}\right)$$

The displacement is then

$$u(t) = \sqrt{\frac{101}{2500}} \cos\left(5t - \arctan\left(-\frac{1}{10}\right)\right) + \frac{1}{3}t \sin(5t).$$

□

2. Forced, Damped Motion

The final case to consider occurs when we have both forcing and damping. Our system has all the forces we listed in the first vibrations lecture acting on it, so the equation becomes

$$mu'' + \gamma u' + ku = F(t).$$

Again, we'll assume for the purposes of this discussion that $F(t)$ is sinusoidal, but this isn't necessarily going to be the case in practice. We won't go into too much detail here, but notice that our general solution, once again, has the form

$$u(t) = u_c(t) + U_p(t).$$

The complimentary solution $u_c(t)$ is just the solution to the associated free, damped case, and it will have different forms depending on whether the system is overdamped, underdamped, or critically damped. Regardless, we know from our discussion of free, damped motion that there will be an exponential involved, and as a result, $\lim_{t \rightarrow \infty} u(t) = 0$.

Since every term of $u_c(t)$ has an exponential, and our forcing function is of the form

$$F(t) = F_0 \cos(\omega t) \quad \text{or} \quad F(t) = F_0 \sin(\omega t),$$

our initial guess for the particular solution of

$$U_p(t) = A \cos(\omega t) + B \sin(\omega t)$$

will be correct. As a result, the general solution will look like

$$u(t) = u_c(t) + A \cos(\omega t) + B \sin(\omega t)$$

$$= u_c(t) + R \cos(\omega t - \delta).$$

Since we know $u_c(t) \rightarrow 0$ we call $u_c(t)$ the *transient solution*. $U_p(t)$ will persist (and continue to oscillate with constant amplitude R), and in the limit, $u(t) \rightarrow U_p(t)$. We call $U_p(t)$ the *steady-state*

solution. Thus, our solution will initially look like the sum of these two functions, but eventually u_c dies out and the displacement looks just like U_p , which comes from the forcing.

Observe that with this choice of $F(t)$, we won't get any resonance as we could have in the undamped case, even if $\omega = \omega_0$. If our system is underdamped, we might get a beating phenomenon, but it will die out as the transient solution vanishes and the steady-state solution takes over.

EXERCISE. Assume our system is underdamped. If $F(t) = F_0 e^{-\alpha t} \cos(\beta t)$, where $\alpha \pm i\beta$ are the roots of the characteristic equation of

$$mu'' + \gamma u' + ku,$$

our guess at the form of $U_p(t)$ will need to be adjusted. What will the displacement function $u(t)$ look like in this case, and what happens to it as $t \rightarrow \infty$? Do we have a similar decomposition into a transient solution and a steady-state solution, or will it vanish? Is there an analogous phenomenon to resonance in this case?