

LECTURE 20

## Good (Mechanical) Vibrations

### 1. Recap

Last class, we saw how to derive a differential equation for a spring-mass system from Newton's Second Law. The equation we derived was

$$(20.1) \quad mu'' + \gamma u' + ku = F(t)$$

where  $u(t)$  is the displacement of the object at time  $t$  (with downward displacement being positive),  $m$  is the object's mass,  $\gamma \leq 0$  is the damping coefficient (0 if there is no damping,  $> 0$  if there is),  $k > 0$  is the spring coefficient, and  $F(t)$  is the sum of any external forces we may have present in our system.

It's critical to remember the sign conventions we're using for writing down initial conditions: downward displacements and forces are positive, upward ones are negative.

We also considered the case of free, undamped motion, where there are no external forces nor any damping. In this case, the equation becomes

$$mu'' + ku = 0.$$

We saw that this has solution

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t),$$

where  $\omega_0 = \sqrt{km}$  is referred to as the *natural frequency* of the system, since it is the frequency the system will oscillate at in the absence of any damping or external factors. We also saw that we can rewrite this equation as a single cosine, in the form

$$u(t) = R \cos(\omega_0 t - \delta),$$

where  $R = \sqrt{c_1^2 + c_2^2}$  is the *amplitude of displacement* and  $\delta$ , given by  $\tan(\delta) = \frac{c_2}{c_1}$ , is the *phase angle of displacement*. This form is preferable in many situations, because it lets us know, at a glance, what the amplitude and any phase shifting are, whereas the previous form only readily gives us the frequency. The general strategy here is to calculate the constants  $c_1$  and  $c_2$  using the initial conditions in the first form, then convert to the second, more physically revealing, form.

### 2. Free, Damped Motion

Now, let's consider what happens if we add a damper into the system with damping coefficient  $\gamma$ . We're still considering free motion, so  $F(t) = 0$ , and our differential equation becomes

$$mu'' + \gamma u' + ku = 0.$$

The characteristic equation is

$$mr^2 + \gamma r + k = 0,$$

and this has solution

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$

There are three different cases we need to consider, corresponding to the discriminant being positive, zero, or negative.

(1)  $\gamma^2 - 4mk = 0$

This case gives us a double root of  $r = -\frac{\gamma}{2m}$ , and so the general solution to our equation is

$$u(t) = c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t}.$$

Notice that  $\lim_{t \rightarrow \infty} u(t) = 0$ , which is good, since that's the whole point of damping.

This case is called *critical damping*, and it occurs when

$$\begin{aligned}\gamma^2 - 4mk &= 0 \\ \gamma &= \sqrt{4mk} = 2\sqrt{mk}\end{aligned}$$

This value of  $\gamma = 2\sqrt{mk}$  is denoted by  $\gamma_{CR}$  and is called the *critical damping coefficient*. Since this case separates the other two (which have very different behaviors, as we will see), it's generally useful to be able to calculate this coefficient for a given spring-mass system, which we can do using this formula.

Critically damped systems may cross  $u = 0$  once (depending on the coefficients  $c_1$  and  $c_2$ ), but they will never cross it more than that. Their general behavior is for the motion to settle back to equilibrium.

(2)  $\gamma^2 - 4mk > 0$

In this case, the discriminant is positive and so we will get two distinct, real roots  $r_1$  and  $r_2$ . Hence our general solution is

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

But what is the behavior of this solution? The solution *should* die out, since we have damping, but we'll need to be sure of this.

We want to be sure that in this case, we still have  $\lim_{t \rightarrow \infty} u(t) = 0$ . To do this, we'll need to rewrite our roots a little.

$$\begin{aligned}r_{1,2} &= \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} \\ &= \frac{-\gamma \pm \gamma \left( \sqrt{1 - \frac{4mk}{\gamma^2}} \right)}{2m} \\ &= -\frac{\gamma}{2m} \left( 1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}} \right)\end{aligned}$$

By assumption, we have that  $\gamma^2 > 4mk$ . Hence

$$1 - \frac{4mk}{\gamma^2} < 1$$

and so

$$\sqrt{1 - \frac{4mk}{\gamma^2}} < 1.$$

So, the quantity in parentheses above is guaranteed to be positive, which means both of our roots will be negative.

Thus the damping in this case has the desired effect, and the vibration will die out in the limit. This case, which occurs when  $\gamma > \gamma_{CR}$ , is called *overdamping*.

The general behavior here is similar to that in the critically damped case. The solution won't oscillate around equilibrium, but rather, it will just settle back into place.

$$(3) \quad \gamma^2 < 4mk$$

The final case occurs when  $\gamma < \gamma_{CR}$ . In this case, the characteristic equation has complex roots

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \alpha + i\beta.$$

The displacement is

$$\begin{aligned} u(t) &= c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t) \\ &= e^{\alpha t} (c_1 \cos(\beta t) + c_2 \sin(\beta t)). \end{aligned}$$

In analogy to the free, undamped case, we can write this as

$$u(t) = R e^{\alpha t} \cos(\beta t - \delta).$$

We know  $\alpha < 0$ . Hence the displacement in this case will also settle back to equilibrium. There is a difference, though: solutions in this case will oscillate even as the oscillations have smaller and smaller amplitude. This case is called *underdamped*.

Notice that the solution  $u(t)$  isn't quite periodic. It has the form of a cosine, but the amplitude isn't constant. A function  $u(t)$  is called *quasi-periodic*, since it oscillates with a constant "frequency" but has varying amplitude.  $\beta$  is called the *quasi-frequency* of the oscillation.

So when we have free, damped vibrations, there are three cases. There's underdamping, where the damping is there, so the vibrations die out, but the system still oscillates around the equilibrium. Mathematically, this corresponds to the characteristic equation having complex roots. Then, there's overdamping, where there's enough damping to suppress oscillation. Here, we have distinct real roots of the characteristic equation. The separating case, critical damping, occurs for a particular damping coefficient,  $\gamma_{CR} = 2\sqrt{mk}$ , that makes the discriminant of the characteristic equation zero. In this case, like in the overdamped case, there is no oscillation.

A good example to keep in mind when considering the types of damping are a car's shocks. If your shocks are new, your car is likely overdamped: when you hit a bump in the road, the car settles back into place. As the car's shocks wear, there's more of an initial jostle, but the car still doesn't bounce around. Eventually, though, your car behaves more like a beat up Volkswagon: when you hit a bump, your car bounces up and down for a few minutes, though it eventually settles.

This last case corresponds to the car's shocks only underdamping the car's motion. The previous case (generally) corresponds to the overdamped case, and the critical point where the car goes from settling to bouncing is the critically damped case.

Another example to keep in mind is that of a washing machine: new washing machines don't vibrate significantly due to the presence of good dampers. Old (or cheap) washing machines, on the other hand, vibrate quite a lot.

In practice, we want to avoid underdamping. We don't want our cars to bounce around on the road and we don't want the top of tall buildings to sway in the wind. With critical damping, we have the right sort of behavior, but it's too precise a case: if our dampers wear even a little we're in the underdamped case. Thus, in practice, we want to be overdamped, so there's some margin for the dampers to wear without going into the overdamped case. Oscillations are generally good things in the real world.

Now, let's do some examples of damped vibrations.

**EXAMPLE 20.1.** *A 2kg object stretches a spring by  $\frac{5}{8}$  m. A damper is attached that exerts a resistive force of 48 N when the speed is 3 m/sec. If the initial displacement is 1 m upwards and the initial velocity is 2 m downwards, find the displacement  $u(t)$  at any time  $t$ .*

This is actually the example from last class with a damper attached and slightly different initial conditions. We've already calculated the spring coefficient  $k = 32$ . What is the damping coefficient

$\gamma$ ? We know that  $|F_d| = 48$  when the speed  $|u'| = 3$ . So the damping coefficient is given by

$$\gamma = \frac{|F_d|}{|u'|} = \frac{48}{3} = 16.$$

Thus the initial value problem is

$$2u'' + 16u' + 32u = 0 \quad u(0) = -1 \quad u'(0) = 2.$$

Before we solve it, let's see which case we're in. To do so, let's calculate the critical damping coefficient.

$$\gamma_{CR} = 2\sqrt{mk} = 2\sqrt{64} = 16.$$

So we're critically damped. This means we should get a double root. Surely enough, the characteristic equation has the double root  $r = -4$ , and so the general solution is

$$u(t) = c_1e^{-4t} + c_2te^{-4t}.$$

The initial conditions give coefficients  $c_1 = -1$  and  $c_2 = -2$ . So the solution is

$$u(t) = -e^{-4t} - 2te^{-4t}.$$

Again, notice the lack of oscillation in this case. □

**EXAMPLE 20.2.** *For the same spring-mass system as in the previous example, attach a damper that exerts a force of 40 N when the speed is 2 m/s. Find the displacement at any time  $t$ .*

The only difference between this example and the previous one is the damping force. Let's compute the damping coefficient.

$$\gamma = \frac{|F_d|}{|u'|} = \frac{40}{2} = 20.$$

Since we computed  $\gamma_{CR} = 16$ , this means we are overdamped and the characteristic equation should give us distinct real roots. The initial value problem is

$$2u'' + 20u' + 32u = 0 \quad u(0) = -1 \quad u'(0) = 2.$$

The characteristic equation has roots  $r_1 = -8$  and  $r_2 = -2$ . So the general solution is

$$u(t) = c_1e^{-8t} + c_2e^{-2t}.$$

The initial conditions give  $c_1 = 0$  and  $c_2 = -1$ , so the displacement is

$$u(t) = -e^{-2t}.$$

Notice here again, we don't actually have a "vibration" as we normally think of them. The damper is strong enough to force the vibrations to die out so quickly that we don't notice much, if any, of them. □

**EXAMPLE 20.3.** *For the same spring-mass system as in the previous two examples, add a damper that exerts a force of 16 N when the speed is 2 m/s.*

In this case, the damping coefficient is

$$\gamma = \frac{16}{2} = 8,$$

which tells us that this case is underdamped as  $\gamma < \gamma_{CR} = 16$ . We should expect to get complex roots of the characteristic equation. The initial value problem is

$$2u'' + 8u' + 32u = 0 \quad u(0) = -1 \quad u'(0) = 3.$$

The characteristic equation has roots

$$r_{1,2} = \frac{-8 \pm \sqrt{192}}{4} = -2 \pm i\sqrt{12}.$$

Thus our general solution is

$$u(t) = c_1 e^{-2t} \cos(\sqrt{12}t) + c_2 e^{2t} (\sqrt{12}t).$$

The initial conditions give constants  $c_1 = -1$  and  $c_2 = \frac{1}{\sqrt{12}}$ , so we have

$$u(t) = -e^{-2t} \cos(\sqrt{12}t) + \frac{1}{\sqrt{12}} e^{2t} (\sqrt{12}t).$$

Let's write this as a single cosine.

$$R = \sqrt{(-1)^2 + \left(\frac{1}{\sqrt{12}}\right)^2} = \sqrt{\frac{13}{12}}$$
$$\tan(\delta) = -\frac{1}{\sqrt{12}}$$

As in the undamped case, we look at the signs of  $c_1$  and  $c_2$  to figure out which quadrant  $\delta$  is in. By doing so, we see that  $\delta$  has [negative cosine and positive sine, so it's in Quadrant II. Hence we need to take the arctangent and add  $\pi$  to it.

$$\delta = \arctan\left(-\frac{1}{\sqrt{12}}\right) + \pi.$$

Thus our displacement is

$$u(t) = \sqrt{\frac{13}{12}} e^{-2t} \cos\left(\sqrt{12}t - \arctan\left(-\frac{1}{\sqrt{12}}\right) - \pi\right).$$

In this case, we actually get a vibration, even though its amplitude steadily decreases until it becomes negligible. The vibration has quasifrequency  $\sqrt{12}$ .  $\square$