LECTURE 19

Mechanical Vibrations

1. An Application!

We're finally at a point where we can stop discussing solutions and properties of second order linear equations. Now, we'll turn our attention to an application: mechanical vibrations. The particular application we'll be considering is an object of given mass m hanging from a spring of natural length l, but there are a number of applications in various branches of engineering that only differ from our setup by some specifics and notation.

Our convention is *always* that downward displacements and forces are *positive*, while upward displacements and forces are negative. It's important to be consistent. We also measure all displacements of the object from its equilibrium position. Thus, if our displacement function is $u(y)$, $u = 0$ corresponds to the center of gravity as it hangs at rest from the spring.

The first task is to develop a differential equation to model the displacement u of the object. First, recall Newton's Second Law

$$ma = F,$$

where m is the mass of the object. We want our equation to be for displacement, so we'll replace a by u'', and Newton's Second Law becomes

$$mu'' = F(t, u, u').$$

What are the various forces acting on the object? We will consider four different forces, some of which may not act on the system in every situation.

(1) **Gravity,** F_g

The gravitational force always acts on the object. It's given by

$$F_g = mg,$$

where g is the acceleration due to gravity. For simpler computations, we'll take $g = 10\text{m/s}$. Notice that this force is positive, since it always acts downward.

(2) **Spring,** F_s

Since we're attaching the object to a spring, the spring will itself always exert a force on the object. We'll assume Hooke's Law governs this force. Hooke's Law says that the spring force is proportional to the displacement of the spring from its natural length. What is the displacement of the spring? When we attach an object to a spring, the spring gets stretched. Let's denote the length of the stretched spring by L. Then its displacement from its natural length is $L + u$.

So, the spring force is given by

$$F_s = -k(L + u),$$

where $k > 0$ is the *spring constant*. Why is that negative there? It ensures that the force has the correct direction. If $u > -L$, *i.e.*, the spring has been stretched beyond its natural length, then $u + L > 0$ and so $F_s < 0$, which is good because we would expect the spring to pull upward on the object in this situation. If $u < -L$, so that the spring is compressed, the spring force would push the object back downwards, and so we expect to find $F_s > 0$. Surely enough, that's what we get.

(3) **Damping,** F_d

We will consider some situations where a damper is attached to the system. They will not always be present, but we'll always note when there is a damper involved. Dampers work to counteract motion (a good example of one is the shocks on your car), so that they will always oppose the direction of the object's velocity.

In other words, if the object has downward velocity $u' > 0$, we'd want the damping force to be acting in the upwards direction, so that $F_d < 0$. Similarly, if $u' < 0$, we want $F_d > 0$. We'll further assume that all of our dampers are linear. So we end up with

$$F_d = -\gamma u',$$

where $\gamma > 0$ is the *damping coefficient.*

(4) **External Forces,** $F(t)$

This is more of a catchall than a particular force. Sometimes we'll want our spring-mass system to have some external forces acting on it (for example, we might hook our system up to a generator which will exert some additional force on it0. We call $F(t)$ the *forcing function*, and it's just the sum of any of these external forces we have in a particular situation.

It's important to reiterate here that in a given problem, all of these forces may not necessarily act on the spring-mass system. So our force function will change depending on the particular situation. When actually writing down our differential equation for a particular situation, we'll need to be aware of what forces are and are not present. But let's go ahead and write down the most general form of our equation, and we'll discuss how it changes in particular cases.

We have $F(t, u, u') = F_g + F_s + F_d + F(t)$, so that Newton's Second Law becomes

$$mu'' = mg - k(L + u) - \gamma u' + F(t),$$

or, upon reordering this a little,

$$mu'' + \gamma u' + ku = mg - kL + F(t).$$

Let's now think about what happens when the object is at rest. When the object is at equilibrium $u = 0$, there are only two forces acting on the object: gravity and the spring force. Since the object is at rest, these two forces must be balancing each other out: $F_g + F_s = 0$. In other words,

$$mg = kL.$$

So our equation simplifies to

$$mu'' + \gamma u' + ku = F(t), \tag{19.1}$$

and this is the most general form of our equation, with all forces present. Along with this differential equation, we'll have initial conditions of the form

$u(0) = u_0$	Initial displacement from the equilibrium position
$u'(0) = u'_0$	Initial velocity.

It's important to keep our sign conventions in mind when writing these down.

Before we start talking about particular cases, we need to discuss how we might go about figuring out these constants k and γ if they're not explicitly given to us in a problem. Let's start with the spring constant k. We know that if the spring is attached to some object with mass m, the object stretches the spring by some length L when it's at rest. We observed above that at equilibrium, $mg = kL$. Thus, if we know how much some object with a known mass stretches the spring when it's at rest, we can compute

$$k = \frac{mg}{L}.$$

This may not necessarily be the same object as the one in the spring-mass system, but that doesn't matter.

How do we compute γ? If we don't explicitly know the damping coefficient from the beginning, we'll know how much force the damper exerts to oppose motion of a given speed. Then we can set $|F_d| = \gamma|u'|$, where $|F_d|$ is the magnitude of the damping force and $|u'|$ is the speed of motion. So we have $\gamma = \frac{F_d}{u}$. We'll see examples of this computation when we consider damped motion.

Let's start looking at specific cases. These will be defined by which forces are acting on our spring-mass system.

2. Free, Undamped Motion

We'll begin by assuming that there are no dampers or external forces acting on our system. This is the simplest situation, and as we have no damping, we can take $\gamma = 0$. Our differential equation is then just

$$mu'' + ku = 0, \tag{19.2}$$

where $m, k > 0$. This is easy enough to solve. The characteristic equation is

$$mr^2 + k = 0,$$

which has roots

$$r_{1,2} = \pm i\sqrt{\frac{k}{m}}.$$

We'll write

$$r_{1,2} = \pm i\omega_0,$$

where we've substituted

$$\omega_0 = \sqrt{k}m.$$

ω_0 is called the *natural frequency* of the system, for reasons that will be clear shortly.

Since the roots of our characteristic equation are imaginary, the form of our general solution is

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t). \tag{19.3}$$

This is why we called ω_0 the natural frequency of the system: it's the frequency of motion when the spring-mass system has no interference from dampers or external forces.

This is all well and good: we know it's a general solution to our equation and it will easily let us solve for the constants c_1 and c_2 given initial conditions. It's not an ideal form for the solution, though, since there's important physical information that we can't necessarily obtain just from looking at it. We can't immediately recover the amplitude of the motion, for example. So what we will sometimes want to do, after we've solved Equation 19.3 for the constants, is to rewrite Equation 19.3 as

$$u(t) = R\cos(\omega_0 t - \delta), \tag{19.4}$$

where $R > 0$ is the *amplitude of displacement* and δ is the *phase angle of displacement*, sometimes referred to as the *phase shift*.

Before we talk about how to rewrite Equation 19.3 as Equation 19.4, let's discuss briefly the pros and cons of both forms. When the displacement is written as Equation 19.3, it's a lot easier to find the constants of integration. But Equation 19.4 is easier to work with: we can directly read off quantities like the amplitude, and it's a lot easier to graph Equation 19.4 than it is to graph Equation 19.3. So the ideal situation would be for us to solve Equation 19.2 in the form of Equation 19.3, find the constants c_1 and c_2 using initial conditions, and then convert to Equation 19.4.

Let's assume that, for a given problem, we've found c_1 and c_2. How do we compute R and δ? Consider Equation 19.4. Using a trig identity, we can write it as

$$u(t) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t). \tag{19.5}$$

Comparing Equation 19.5 to Equation 19.3, we see that

$$c_1 = R\cos(\delta) \qquad c_2 = R\sin(\delta).$$

Notice

$$c_1^2 + c_2^2 = R^2(\cos^2(\delta) + \sin^2(\delta)) = R^2,$$

so that, assuming $R > 0$,

$$R = \sqrt{c_1^2 + c_2^2}.$$

Now, let's consider the quantity

$$\frac{c_2}{c_1} = \frac{\sin(\delta)}{\cos(\delta)} = \tan(\delta).$$

Some care will be needed here, though, due to the fact that the range of arctan is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So we can't just take the arctan of both sides, since that would always result in our angle δ being in Quadrants I or IV. What we'll need to do, then, is determine what quadrant δ should be in by considering its sine, c_1, and its cosine, c_2. At that point, we can solve

$$\tan(\delta) = \frac{c_2}{c_1}$$

for the correct angle δ. We'll see examples of this shortly.

Let's do an example.

Example 19.1. *A 2kg object is attached to a spring, which it stretches by $\frac{5}{8}$ m. The object is given an initial displacement of 1 m upwards and given an initial downwards velocity of 4 m/sec. Assuming there are no other forces acting on the spring-mass system, find the displacement of the object at any time t and express it as a single cosine.*

The first step is to write down the initial value problem for this setup. We'll need to find m and k. m is easy: we know the mass of the object is 2kg. How about k? From our earlier discussion, we know that

$$k = \frac{mg}{L} = \frac{(2)(10)}{\frac{5}{8}} = 32.$$

So our differential equation is

$$2u'' + 32u = 0.$$

The initial conditions are given by

$$u(0) = -1 \qquad u'(0) = 4.$$

The characteristic equation here is

$$2r^2 + 32 = 0,$$

and this has roots $r_{1,2} = \pm 4i$. Hence $\omega_0 = 4$, and our general solution is

$$u(t) = c_1\cos(4t) + c_2\sin(4t).$$

Using our initial conditions, we see

$$\begin{aligned} -1 &= u(0) = c_1 \\ 4 &= u'(0) = 4c_2 \Rightarrow c_2 = 1. \end{aligned}$$

So the particular solution in this case is

$$u(t) = \cos(4t) + \sin(4t).$$

We want to write this as a single cosine, since that will make it easier to visualize the motion in question. We start by computing R:

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{2}.$$

Next, let's consider δ. We know

$$\tan(\delta) = \frac{c_2}{c_1} = -1.$$

This means δ is either in Quadrants II or IV. To decide which, we need to look at the values of $\sin(\delta)$ and $\cos(\delta)$. We have

$$\sin(\delta) = c_2 > 0$$
$$\cos(\delta) = c_1 < 0.$$

Our conclusion is that δ is in Quadrant II. If we take $\arctan(-1) = -\frac{\pi}{4}$, this has a value in Quadrant IV. Since tan is π–periodic, however, $-\frac{\pi}{4} + \pi = \frac{3\pi}{4}$ is in Quadrant II and also has a tangent of -1. Thus our desired phase angle is

$$\delta = \arctan\left(\frac{c_2}{c_1}\right) + \pi = \arctan(-1) + \pi = \frac{3\pi}{4},$$

and our solution in the form of Equation 19.4 is

$$u(t) = \sqrt{2}\cos\left(4t - \frac{3\pi}{4}\right).$$

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