LECTURE 7

Existence and Uniqueness

At the beginning of the course, we made a point of noting three questions we wanted to keep in mind:

- *Given an initial value problem, does a solution exist?*
- *If a solution exists, is it unique?*
- *If a solution exists, how do we find it?*

So far, we've been mostly concerned with the last question. Today, we'll discuss the first two: without solving an initial value problem, what information can we derive about existence and uniqueness of solutions? We'll also see some strong differences between linear and nonlinear equations.

1. Linear Equations

While we'll specifically be dealing with first order linear equations in this section, the same basic method works for higher order linear equations as well.

Theorem 7.1 (Linear Fundamental Theorem of Existence and Uniqueness). *Consider the IVP*

$$y' + p(t)y = q(t) \qquad y(t_0) = y_0.$$

If $p(t)$ and $q(t)$ are continuous functions on an open interval $\alpha < t_0 < \beta$, then there exists a unique solution to the IVP defined on the interval (α, β).

Remark. The same result holds for general linear initial value problems: if we have the IVP

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = g(t) \qquad y(t_0) = y_0, y'(t_0) = y_0', \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)}$$

then if $a_i(t)$ (for $i = 0, \ldots, n-1$) and $g(t)$ are continunous on an open interval $\alpha < t_0 < \beta$, there exists a unique solution to the IVP defined on the interval (α, β).

What exactly does Theorem 7.1 tell us? Two things:

1. If the given linear differential equation is nice enough, not only do we know that a solution exists, but we know it will be unique. In a lot of cases, knowing that a unique solution exists is more important than actually having the solution.
2. If the interval (α, β) is the largest interval on which $p(t)$ and $q(t)$ are continuous, then (α, β) is the interval of validity to the unique solution guaranteed by the theorem. Thus, given an initial value problem involving a sufficiently nice linear differential equation, there is no need to solve the differential equation to get the interval of validity. The interval of validity also depends only on t_0 (since the interval must contain it) but not at all on y_0.

Example 7.1. *Without solving, determine the interval of validity for the solution ot the following initial value problem.*

$$(t^2 - 9)y' + 2y = \ln|20 - 4t| \qquad y(4) = -3$$

If we look at Theorem 7.1, we may notice that to use it, we'll need to write the differential equation in the form given in the theorem (*i.e.*, with the coefficient of y' being 1). This gives us

$$y' + \frac{2}{t^2 - 9} = \frac{\ln|20 - 4t|}{t^2 - 9}.$$

Next, we want to find the points where either of the two coefficient functions are discontinuous. By throwing out those points, we can find all of the possible intervals of validity for solutions to the differential equation. Then we can see which one contains t_0.

Using the notation from Theorem 7.1, we have that $p(t)$ is discontinuous at $t = \pm 3$, since at those values we're dividing by zero. $q(t)$ is discontinuous at $t = \pm 3$ for the same reason while also being discontinuous at $t = 5$, where we would be taking the logarithm of zero (notice we don't need to worry about taking logarithms of negative numbers because of the absolute value signs).

This yields four intervals on which both $p(t)$ and $q(t)$ are continuous:

$$(-\infty, -3) \qquad (-3, 3) \qquad (3, 5) \qquad (5, \infty).$$

Notice that the endpoints of each of these intervals are points where either $p(t)$ or $q(t)$ are discontinuous. Hence we can guarantee that both will be nice on each interval.

Now we need to identify which of these intervals of the interval of validity for the solution to our particular initial value problem. The interval must contain $t_0 = 4$. Hence the interval of validity will be $(3, 5)$. □

Remark. The other intervals we found in the previous example could be intervals of validity for the same differential equation with a different initial condition. For example, if our initial condition had been $y(2) = 5$, we would have concluded that the unique solution to the initial value problem has an interval of validity of $(-3, 3)$.

What happens if the initial condition is at one of these bad values where either $p(t)$ or $q(t)$ are discontinuous? The answer is that we can't conclude anything. If the hypotheses of the theorem aren't met, then it's not true that the conclusions of the theorem are false. We just can't conclude anything. It may be that the solution doesn't exist or that infinitely many solutions exist. We just don't know.

Let's do one more example.

Example 7.2. *Without solving, find the interval of validity to the following initial value problem.*

$$\cos(x)y' = \sin(x)y - \sqrt{x-1} \qquad y\left(\frac{3}{2}\right) = 0$$

First, we need to put the equation into the proper form:

$$y' - \tan(x)y = -\frac{\sqrt{x-1}}{\tan(x)}.$$

Using the notation from Theorem 7.1, we have that $p(t)$ is discontinuous at $x = \frac{n\pi}{2}$ for odd integers n and $q(t)$ is discontinuous there and for any $x < 1$. Thus we can list the possible intervals of validity.

$$\left(1, \frac{\pi}{2}\right) \qquad \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \qquad \cdots \qquad \left(\frac{(2n+1)\pi}{2}, \frac{(2n+3)\pi}{2}\right)$$

for all positive integers n. Since our interval of validity is $y\left(\frac{3}{2}\right) = 0$, we can conclude that the interval of validity is $\left(1, \frac{\pi}{2}\right)$. □

2. Nonlinear Equations

Things get more tricky with nonlinear equations. In the linear case, every "nice enough" equation had a unique solution except for a few bad initial conditions. But even the innocent looking nonlinear equation

$$\left(\frac{dy}{dx}\right)^2 + x^2 + 1 = 0$$

has no real solutions.

Now, let's look at the more general version of the theorem, which applies to nonlinear equations. It has a much weaker conclusion than the previous one, and we won't be spending too much time on it, but it's important to understand.

Theorem 7.2. *Consider the IVP*

$$y' = f(t, y) \qquad y(t_0) = y_0.$$

If f and $\frac{\partial f}{\partial y}$ are continuous functions on some rectangle $\alpha < t_0 < \beta$, $\gamma < y_0 < \delta$ containing the point (t_0, y_0), then there is a unique solution to the initial value problem defined on some interval (a, b) satisfying $\alpha \le a < t_0 < b \le \beta$.

There are a few things to notice here.

1. Unlike Theorem 7.1, Theorem 7.2 doesn't tell us the interval of validity for a unique solution guaranteed by it. Instead, it tells us the largest possible interval that the solution will exist in; we'd need to actually solve the IVP to get the interval of validity.
2. For nonlinear differential equations, the value of y_0 may affect the interval of validity, as we will see in a later example. The general strategy is to make sure that our initial condition doesn't lie in or on the boundary of a "bad" region (where either f or its derivative are discontinuous) and then find the largest t-interval on the line $y = y_0$ containing t_0 where everything is nice and continuous.

Remark. Theorem 7.2 refers to the *partial derivative* $\frac{\partial f}{\partial y}$ of the function of two variables $f(t, y)$. We'll talk a bit more about this later. Computationally, finding this is no more difficult than computing ordinary derivatives. We treat t as a constant and differentiate. So, for example, if $f(t, y) = t^2 - 2y^3 t$, $\frac{\partial f}{\partial y} = -6yt$.

Example 7.3. *Determine the largest possible interval of validity for the initial value problem*

$$y' = x \ln(y) \qquad y(2) = e.$$

We have $f(x, y) = x \ln(y)$, so $\frac{\partial f}{\partial y} = \frac{x}{y}$. f As far as y is concerned, the areas to be avoided are $y \le 0$. Since our initial condition is $y = e > 0$, this is no problem. Now, there are no discontinuities involving x for either, so we will have a unique solution that exists somewhere inside $(-\infty, \infty)$. □

Example 7.4. *Determine the largest possible interval of validity for the initial value problem*

$$y' = \sqrt{y - t^2} \qquad y(0) = 1.$$

$f(t, y) = \sqrt{y - t^2}$ and $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y - t^2}}$. The region of discontinuities is given by $y \le t^2$. Our initial condition $y(0) = 1$ doesn't lie in this region, so we can continue. The line $y = 1$ contains discontinuities at $-1 < t < 1$, so our conclusion is that the interval of validity of the guaranteed unique solution is contained somewhere within $(-1, 1)$. □

What can happen when the conditions of Theorem 7.2 aren't met?

Example 7.5. *Determine all possible solutions to the IVP*

$$y' = y^{\frac{1}{3}} \qquad y(0) = 0.$$

Let's first note that this does not satisfy the conditions of the theorem, since $\frac{\partial f}{\partial y} = \frac{1}{3y^{\frac{2}{3}}}$ is not continuous at the origin. Ok, so let's solve this. It's separable, and we first note that we have an equilibrium solution $y(0) = 0$. This is one that satisfies the initial conditions, but let's continue on.

$$\int y^{-\frac{1}{3}}\,dy = \int dt$$
$$\frac{3}{2}y^{\frac{2}{3}} = t + c$$

Using the initial condition gives $c = 0$, so we get that the solution is

$$y(t) = \pm\left(\frac{2}{3}t\right)^{\frac{3}{2}}.$$

Notice that using our initial condition doesn't allow us to choose between these possibilities, since both are satisfied. Thus we end up with three possible solutions to the initial value problem.

□

This is a good reminder that things don't always work nicely in the world of differential equations. In this class, most of the problems we'll work will be nice and have unique solutions, but this isn't always the case when one actually works with differential equations. Now, let's look at a case that illustrates the pontential dependance on y_0 of the interval of validity to a solution in the nonlinear case.

Example 7.6. *Determine the interval of validity for the initial value problem*

$$y' = y^2 \qquad y(0) = y_0.$$

First, we notice that this is a nonlinear equation. It's also not hard to see that it meets the conditions of Theorem 7.2 for every value of y_0, and so regardless of the choice of y_0 there will be a unique solution. The theorem tells us that the solution will be defined somewhere inside $(-\infty, \infty)$, which is very helpful indeed.

So let's solve. First, note that if $y_0 = 0$, the solution is the equilibrium solution $y(0) = 0$. Now, let's assume $y_0 \neq 0$.

$$\int y^{-2}\,dy = \int dt$$
$$-\frac{1}{y} = t + c$$

Applying the initial condition gives

$$c = -\frac{1}{y_0}.$$
$$-\frac{1}{y} = t - \frac{1}{y_0}$$
$$y(t) = \frac{y_0}{1 - y_0 t}$$

What is the interval of validity here? The only point of discontinuity is $t = \frac{1}{y_0}$. So the two possible intervals of validity are

$$\left(-\infty, \frac{1}{y_0}\right) \qquad \left(\frac{1}{y_0}, \infty\right).$$

The correct choice will be the interval that contains $t_0 = 0$. But this will depend on y_0. If $y_0 > 0$, 0 will be contained in the interval $\left(-\infty, \frac{1}{y_0}\right)$ and so this is the interval of validity. On the other hand, if $y_0 < 0$, 0 is contained inside $\left(\frac{1}{y_0}, \infty\right)$ and so this is the interval of validity. Thus we have the following possible intervals of validity, depending on what y_0 is.

If $y_0 > 0$ $\left(-\infty, \frac{1}{y_0}\right)$ is the interval of validity .
If $y_0 = 0$ $(-\infty, \infty)$ is the interval of validity. □
If $y_0 < 0$ $\left(-\frac{1}{y_0}, \infty\right)$ is the interval of validity.

3. Summary

What did we learn from this section? We saw the conditions for existence of a unique solution to an initial value problem involving a first order equation (and for higher order linear equations, in fact). Intervals of validity for linear equations don't depend on the initial choice of y_0, while they may for nonlinear equations.

Second, we can find intervals of validity for solutions for linear equations without having to solve the equation, which is very useful. For a nonlinear equation, on the other hand, we would actually need to solve the equation. All we can potential find are places where the unique solution will definitely not be defined.