LECTURE 18

Undetermined Coefficients: The Revenge

1. This Is It For Undetermined Coefficients, I Promise

We ended last class by noticing that if part of our guess for a particular solution $Y_p(t)$ coincided with a bit of the complimentary solution, we needed to multiply the offending part by t to ensure that it doesn't vanish when we plug it into the equation. Let's look at some examples.

EXAMPLE 18.1. *Write down a guess for the form of a particular solution to the following differential equations.*

(1) $y'' - 3y' - 28y = 6t + e^{-4t} - 2$

First, we find the complimentary solution. It is

$$y_c(t) = c_1 e^{7t} + c_2 e^{-4t}.$$

Our nonhomogeneous term is $g(t) = 6t + e^{-4t} - 2$, which we can rearrange to group the polynomial terms together as $g(t) = 6t - 2 + e^{-4t}$. Thus our initial guess would be

$$At + B + Ce^{-4t}.$$

The first two terms aren't a problem, but the Ce^{-4t} term also appears in the complimentary solution. Since Cte^{-4t} doesn't show up in the complimentary solution, our final guess is

$$Y_p(t) = At + B + Cte^{-4t}.$$

(2) $y'' - 64y = t^2 e^{8t} + \cos(t)$

The complimentary solution is

$$y_c(t) = c_1 e^{8t} + c_2 e^{-8t}.$$

Our initial guess for a particular solution is

$$(At + Bt + C)e^{8t} + D\cos(t) + E\sin(t).$$

If we distributed the exponential through the polynomial, we'd have a Ce^{8t} term which also showed up in our complimentary solution. What we'll need to do is to multiply the *entire* first term by t (to see why, just differentiate and plug in...you'll see that if we don't, we'll end up losing a coefficient which we'll need later). So our final guess is

$$Y_p(t) = t(At^2 + Bt + C)e^{8t} + D\cos(t) + E\sin(t).$$

(3) $y'' + 4y' = e^{-t}\cos(2t) + t\sin(2t)$

The complimentary solution is

$$y_c(t) = c_1\cos(2t) + c_2\sin(2t).$$

Our first guess for a particular solution would be

$$e^{-t}(A\cos(2t) + B\sin(2t)) + (Ct + D)\cos(2t) + (Et + F)\sin(2t).$$

First, we notice that despite having similar looking terms, we can't actually combine anything, since the similar terms are multiplied by factors which don't just differ by constant coefficients. Next, we notice that both the second and third terms involve components of the complimentary solution: $D\cos(2t)$ and $F\sin(2t)$. Thus we'll need to multiply those two terms by t. The first term is ok, though, since if we multiplied it out we would have a product of an exponential and a sine or a cosine, and those aren't terms in the complimentary solution. So we end up with

$$Y_p(t) = e^{-t}(A\cos(2t) + B\sin(2t)) + t(Ct + D)\cos(2t) + t(Et + F)\sin(2t).$$

(4) $y'' + 2y' + 5y = e^{-t}\cos(2t) + t\sin(2t)$

Notice that the nonhomogeneous term in this example is the same as in the previous one; we've just changed the differential equation. Here, the complimentary solution is

$$y_c(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t).$$

Our initial guess for the particular solution is the same as in the last example:

$$e^{-t}(A\cos(2t) + B\sin(2t) + (Ct + D)\cos(2t) + (Et + F)\sin(2t).$$

This time, the first term causes the problem, while the second and third are fine just as they are. So we'll multiply the first by t:

$$Y_p(t) = te^{-t}(A\cos(2t) + B\sin(2t)) + (Ct + D)\cos(2t) + (Et + F)\sin(2t).$$

(5) $y'' + 4y' + 4y = t^2 e^{-2t} + 2e^{-2t}$

Here the complimentary solution is

$$y_c(t) = c_1 e^{-2t} + c_2 te^{-2t}.$$

Notice that we can factor out a e^{-2t} from our nonhomogeneous term, which then becomes $g(t) = (t^2 + 2)e^{-2t}$. This is the product of a polynomial and an exponential, so our initial guess is

$$(At^2 + Bt + C)e^{-2t}.$$

But this Ce^{-2t} term is the first term in our complimentary solution. Also, we have Bte^{-2t}, which is the second term in the complimentary solution. So this is no good. Next, we try multiplying by t:

$$t(At^2 + Bt + C)e^{-2t}.$$

This still causes problems: the Cte^{-2t} term is still the second term in our complimentary solution. If we multiply by t^2, though, we have no problems, and so our final guess is

$$Y_p(t) = t^2(At^2 + Bt + C)e^{-2t}.$$

□

So that's about it for Undetermined Coefficients. As long as you're comfortable with the guesses for the basic types, are on the lookout for coefficients that should be combined, and check that you're not replicating any part of the complimentary solution, you can't go wrong.

2. Variation of Parameters

The other major method for determining a particular solution to the linear nonhomogeneous equation

$$p(t)y'' + q(t)y' + r(t)y = g(t)$$

is Variation of Parameters. Undetermined Coefficients works well enough, but the algebra can be quite messy and it only works for a handful of equations.

Variation of Parameters is a much more general method, but it has its drawbacks. We could sometimes do Undetermined Coefficients without the complimentary solution, but this is never possible for Variation of Parameters. Also, while Undetermined Coefficients reduces the problem of finding a particular solution to an algebraic one, Variation of Parameters involves taking some integrals, which we may not always be able to do. So we'll always be able to write down a formula for the solution, but we may not be able to explicitly find it. Still, its generality makes it worth discussing.

How does it work? For this method, we start by dividing through by $p(t)$: we'll want to start with y'' having a coefficient of 1. So the equation we're dealing with is really

$$y'' + q(t)y' + r(t)y = g(t). \tag{18.1}$$

Next, suppose we know the complimentary solution

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t). \tag{18.2}$$

Recall that this is the general solution to the homogeneous equation

$$y'' + q(t)y' + r(t)y = 0.$$

We're going to see if we can find two functions $u_1(t)$ and $u_2(t)$ such that

$$Y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t). \tag{18.3}$$

This is similar in spirit to doing reduction of order: we know that combinations of y_1 and y_2 with constant coefficients gives solutions to Equation 18.2, so the next thing to try is a combination of y_1 and y_2 involving nonconstant functions.

However, to do this, we need to make an assumption. If we differentiate Equation 18.3, we obtain

$$Y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'.$$

The assumption we will make is that

$$u_1' y_1 + u_2' y_2 = 0.$$

Why do we make this assumption? Unfortunately, I don't have a good answer for this. It turns out that it works: it simplifies the expression we'll end up getting so that we can solve it, and it doesn't interfere with obtaining a solution. With this assumption, the first derivative becomes

$$Y_p' = u_1 y_1' + u_2 y_2'.$$

Then the second derivative is

$$Y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'.$$

Let's plug the derivatives into Equation 18.1.

$$\left(u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'\right) + q(t)\left(u_1 y_1' + u_2 y_2'\right) + r(t)\left(u_1 y_1 + u_2 y_2\right) = g(t)$$

Let's rearrange this to take advantage of y_1 and y_2 solving Equation 18.2:

$$\begin{aligned} \left(u_1' y_1' + u_2' y_2'\right) + u_1\left(y_1'' + q(t)y_1' + r(t)y_1\right) + u_2\left(y_2'' + q(t)y_2' + r(t)y_2\right) &= g(t) \\ u_1' y_1' + u_2' y_2' &= g(t). \end{aligned}$$

This is why we wanted the coefficient of y'' to be 1. Otherwise, we would have a fraction on the right hand side of the previous equation, which would in general be nasty.

Remember that we started by supposing that the form of a particular solution was

$$Y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

So we're trying to find these unknown functions $u_1(t)$ and $u_2(t)$. We have two conditions on these functions:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t). \end{aligned}$$

Let's solve this system of equations.

$$\begin{aligned} u_1' &= -\frac{u_2' y_2}{y_1} \\ \left(-\frac{u_2' y_2}{y_1}\right) y_1' + u_2' y_2' &= g(t) \\ u_2' \left(y_2' - \frac{y_2 y_1'}{y_1}\right) &= g(t) \\ u_2' \left(\frac{y_1 y_2' - y_2 y_1'}{y_1}\right) &= g(t) \\ u_2' &= \frac{y_1 g(t)}{y_1 y_2' - y_2 y_1'} \\ u_1' &= -\frac{y_2 g(t)}{y_1 y_2' - y_2 y_1'} \end{aligned}$$

Notice that the quantity in these denominators is the Wronskian of y_1 and y_2, $W(y_1, y_2)$. We know that the Wronskian is nonzero because we started by assuming that we knew y_1 and y_2 were a fundamental set of solutions. By integrating, we get

$$u_1(t) = -\int \frac{y_2 g(t)}{W(y_1, y_2)} dt \qquad u_2(t) = \int \frac{y_1 g(t)}{W(y_1, y_2)} dt.$$

To sum, if we consider the differential equation

$$y'' + q(t)y' + r(t)y = g(t)$$

with complimentary solution

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t),$$

then a particular solution is given by

$$Y_p(t) = -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} dt.$$

Before we do some examples, let's see what will happen to the constants of integration. If we explicitly put them in, we'll get

$$\begin{aligned} Y_p(t) &= -y_1 \left(\int \frac{y_2 g(t)}{W(y_1, y_2)} + c \right) + y_2 \left(\int \frac{y_1 g(t)}{W(y_1, y_2)} + k \right) \\ &= -y_1 \int \frac{y_2 g(t)}{W(y_1, y_2)} + y_2 \int \frac{y_1 g(t)}{W(y_1, y_2)} + (-c y_1 + k y_2). \end{aligned}$$

But we know that this final quantity in parentheses will contribute zero when plugged into Equation 18.1 for any c and k, so it doesn't end up mattering what the constants of integration are. Thus we will assume that they are zero.

Now, let's do an example. In this example, we'll still deal with constant coefficient equations, but you should be aware that this method will, in principle, work for all differential equations of the form (18.1).

Example 18.2. *Find a general solution to the nonhomogeneous equation*

$$y'' - 2y' + y = \frac{e^t}{t^2 + 1}.$$

First, we need the complimentary solution. In this case, it's

$$y_c(t) = c_1 e^t + c_2 t e^t.$$

Thus we have $y_1(t) = e^t$ and $y_2(t) = te^t$. We need the Wronskian of these two functions, which is

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^t(e^t + te^t) - e^t(te^t) = e^{2t}.$$

Then the particular solution is

$$\begin{aligned} Y_p(t) &= -e^t \int \frac{te^t e^t}{e^{2t}(t^2+1)} dt + te^t \int \frac{e^t e^t}{e^{2t}(t^2+1)} dt \\ &= -e^t \int \frac{t}{t^2+1} dt + te^t \int \frac{1}{t^2+1} dt \\ &= -\frac{1}{2} e^t \ln(t^2+1) + te^t \arctan(t). \end{aligned}$$

Thus the general solution is

$$Y(t) = c_1 e^t + c_2 te^t - \frac{1}{2} e^t \ln(t^2+1) + te^t \arctan(t).$$

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