LECTURE 17

Undetermined Coefficients Beyond Thunderdome

1. Recap (again!)

The point of the method of Undetermined Coefficients is to make a guess at the form of a particular solution $Y_p(t)$ of a nonhomogeneous equation based on the form of the nonhomogeneous term $g(t)$. This method works for exponentials (if $g(t) = ae^{\alpha t}$, we guess $Y_p(t) = Ae^{\alpha t}$), trig functions (if $g(t) = a\cos(\alpha t)$ or $a\sin(\alpha t)$, we guess $Y_p(t) = A\cos(\alpha t) + B\sin(\alpha t)$), and polynomials (if $g(t) = a_n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$, we guess $Y_p(t) = A_n t^n + A_{n-1}t^{n-1} + \ldots + A_1 t + A_0$). We saw last class that if $g(t)$ is a product of these basic types, our guess for $Y_p(t)$ is the product of the guesses for these basic types, with care taken to make sure we don't have any extraneous coefficients.

Let's practice writing down the guesses for a couple of these.

Example 17.1. *Write down the form of the particular solution to*

$$y'' - 4y' - 12y = g(t)$$

for the following $g(t)$s:

(1) $g(t) = \left(9t^2 - 103t\right)\cos(t)$

Here we've got the product of a quadratic and a cosine. The guess for the quadratic is

$$At^2 + Bt + C$$

and the guess for the cosine is

$$D\cos(t) + E\sin(t).$$

Multiplying the two guesses gives

$$\begin{gathered}\left(At^2 + Bt + C\right)(D\cos t) + \left(At^2 + Bt + C\right)(E\sin t)\\ \left(ADt^2 + BDt + CD\right)\cos t + \left(AEt^2 + BEt + CE\right)\sin t.\end{gathered}$$

Each of our coefficients here is a product of two constants, which is just another constant. So, as before, to simplify everything, we'll replace each of those with a single constant to yield the following guess.

$$Y_p(t) = \left(At^2 + Bt + C\right)\cos t + \left(Dt^2 + Et + F\right)\sin t.$$

This is indicative of the general rule for a product of a polynomial and a trig function. Write down the guess for the polynomial, multiplied by a cosine, then add to that the guess for the polynomial (with different constants!) multiplied by a sine.

(2) $g(t) = e^{-2t}(3 - 5t)\cos(9t)$

This nonhomogeneous term has all three things. So, combining the two general rules from before, we get

$$Y_p(t) = e^{-2t}(At + B)\cos(9t) + e^{-2t}(Ct + D)\sin(9t).$$

□

2. Sums

We have the following important fact. If Y_1 satisfies

$$p(t)y'' + q(t)y' + r(t)y = g_1(t)$$

and Y_2 satisfies

$$p(t)y'' + q(t)y' + r(t)y = g_2(t)$$

, then $Y_1 + Y_2$ satisfies

$$p(t)y'' + q(t)y' + r(t)y = g_1(t) + g_2(t).$$

This means that if our nonhomogeneous term $g(t)$ is a sum of terms we know how to deal with, we can write down the guesses for each of those terms and add them together for our guess. We may just need to be careful about redundant terms.

Example 17.2. *Find a particular solution to*

$$y'' - 4y' - 12y = e^{7t} + 12.$$

Our nonhomogeneous term $g(t) = e^{7t} + 12$ is the sum of an exponential $g_1(t) = e^{7t})$ and a 0th degree polynomial $g_2(t) = 12$. The guess for $g_1(t)$ is

$$Ae^{7t}$$

while the guess for $g_2(t)$ is

$$B.$$

Adding these together gives

$$Ae^{7t} + B.$$

This can't be simplified in any way, so based on our previous fact we'll go ahead and guess

$$Y_p(t) = Ae^{7t} + B$$

. Differentiating and plugging in yields

$$\begin{aligned} 49Ae^{7t} - 28Ae^{7t} - 12Ae^{7t} - 12B &= e^{7t} + 12 \\ 9Ae^{7t} - 12B &= e^{7t} + 12. \end{aligned}$$

Setting coefficients equal gives $A = \frac{1}{9}$ and $B = -1$, so our particular solution is

$$Y_p(t) = \frac{1}{9}e^{7t} - 1.$$

□

Let's practice writing down some guesses.

Example 17.3. *Write down the form of a particular solution to*

$$y'' - 4y' - 12y = g(t)$$

for each of the following $g(t)$'s:

(1) $g(t) = 2\cos(3t) - 9\sin(3t)$

Our guess for the cosine is

$$A\cos(3t) + B\sin(3t).$$

Additionally, our guess for the sine is

$$C\cos(3t) + D\sin(3t).$$

So if we add the two of them together, we obtain

$$A\cos(3t) + B\sin(3t) + C\cos(3t) + D\sin(3t) = (A + C)\cos(3t) + (B + D)\sin(3t).$$

But $A+C$ and $B+D$ are just some constants, so we can replace them as such as we just yet a guess of

$$Y_p(t) = A\cos(3t) + B\sin(3t).$$

(2) $g(t) = \sin(t) - 2\sin(14t) - 5\cos(14t)$

Again, we start by writing down the guesses for the individual terms. The guess for $\sin(t)$ is

$$A\cos(t) + B\sin(t).$$

Since they have the same argument, the previous example showed us that we can combine the guesses for $2\sin(14t)$ and $-5\cos(14t)$ into

$$C\cos(14t) + D\sin(14t).$$

We won't be able to further combine anything, since the trig functions here have different arguments, and so we end up with a guess of

$$Y_p(t) = A\cos(t) + B\sin(t) + C\cos(14t) + D\sin(14t).$$

(3) $g(t) = 7\sin(10t) - 5t^2 + 4t$

Here we have the sum of a trig function ($7\sin(10t)$ and a quadratic polynomial ($-5t^2+4t$). The guess for the sine is

$$A\cos(10t) + B\sin(10t)$$

while the guess for the quadratic is

$$Ct^2 + Dt + E.$$

Since there's nothing to be consolidated, our guess is

$$Y_p(t) = A\cos(10t) + B\sin(10t) + Ct^2 + Dt + E.$$

(4) $g(t) = 9e^t + 3te^{-5t} - 5e^{-5t}$

If we write down the sum of each of the individual guesses, we would obtain a guess of

$$Ae^t + (Bt + C)e^{-5t} + De^{-5t} = Ae^t + (Bt + (C+D))e^{-5t}.$$

Of course, $C+D$ is just another constant, so our final guess would be

$$Y_p(t) = Ae^t + (Bt + C)e^{-5t}.$$

We could also have noticed that $g(t)$ could be rewritten as

$$g(t) = 9e^t + (3t-5)e^{-5t},$$

in which case writing down the sum of our guesses directly works out to what we just obtained through a slightly more indirect approach.

(5) $g(t) = t^2\sin(t) + 4\cos(t)$.

Writing down the sum of the guesses gives

$$(At^2 + Bt + C)\cos(t) + (Dt^2 + Et + F)\sin(t) + G\cos(t) + H\sin(t),$$

which simplifies to

$$(At^2 + Bt + (C+G))\cos(t) + (Dt^2 + Et + (F+H))\sin(t).$$

Once again, since $C+G$ and $F+H$ are just constants, we end up with

$$Y_p(t) = (At^2 + Bt + C)\cos(t) + (Dt^2 + Et + F)\sin(t).$$

The last two examples have shown us that if we have two terms in our $g(t)$ whose guesses differ only by a polynomial factor, then we can just look for the term corresponding to the highest degree polynomial and the guess for that term will include the guess for the second one. Without making this observation, of course, we can just do what we did here and observe that certain coefficients combine together.

(6) $g(t) = 3e^{-3t} + e^{-3t}\sin(3t) + \cos(3t)$.

We'll need to start here by writing down the sums of each of the guesses:

$$Ae^{-3t} + e^{-3t}(B\cos(3t) + C\sin(3t)) + D\cos(3t) + E\sin(3t).$$

Notice that we can't combine anything, since the various terms that look similar differ by nonconstant factors, not just constant coefficients. So our guess is just

$$Y_p(t) = Ae^{-3t} + e^{-3t}(B\cos(3t) + C\sin(3t)) + D\cos(3t) + E\sin(3t).$$

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As we've seen, sums are quite straightforward to deal with as long as we're on the look out for redundant terms that can be combined.

So, that's all for undetermined coefficients, right? Well...not quite. There's one potential complication that we need to be able to deal with.

EXAMPLE 17.4. *Find a particular solution to*

$$y'' - 4y' - 12y = e^{6t}.$$

This seems straightforward enough: our nonhomogeneous term is an exponential, so we just guess $Y_p(t) = Ae^{6t}$. If we differentiate and plug in, we get

$$\begin{aligned} 36Ae^{6t} - 24Ae^{6t} - 12Ae^{6t} &= e^{6t} \\ 0 &= e^{6t}. \end{aligned}$$

Exponentials are never zero. So this is no good at all. We weren't able to solve for our coefficient, which means we made a mistake in our original guess. But what?

Recall that the complimentary solution in this case is

$$y_c(t) = c_1e^{6t} + c_2e^{-2t}.$$

So our guess is just the first component of our complimentary solution, which is the solution to

$$y'' - 4y' - 12y = 0.$$

As a result, we should have expected that this guess would give us 0 when we plugged it into the left hand side of our equation, since it's a solution to the associated homogeneous equation. So how do we fix this?

Our guess should still involve the exponential e^{6t}, but it can't just be a constant times this exponential. The next thing to try might be $Y_p(t) = Ate^{6t}$, since all we've done to our original guess is to multiply by a linear factor. Let's try it.

$$\begin{aligned} \left(36Ate^{6t} + 12Ae^{6t}\right) - 4\left(6Ate^{6t} + Ae^{6t}\right) - 12Ate^{6t} &= e^{6t} \\ (36A - 24A - 12A)te^{6t} + (12A - 4A)e^{6t} &= e^{6t} \\ 8Ae^{6t} &= e^{6t} \end{aligned}$$

Setting coefficients equal, we conclude that $A = \frac{1}{8}$, so

$$Y_p(t) = \frac{1}{8}te^{6t}.$$

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Notice in Example 17.4 that when we plugged in our new guess, the terms with the t all vanished. This is just a consequence of the product rule: those terms corresponded to differentiating only e^{6t}, and we already saw that those vanish when plugged into the DE. This is why multiplying by t works: the t goes away and the product rule gives us some new terms which don't cancel.

What is the lesson of Example 17.4? We had a problem with our previous guessing method because it led us to guess a term in the complimentary solution. The solution, then, is to find the complimentary solution first, write down guesses, and after comparing the two equations multiply any redundant terms by t.

We do have to be slightly more careful than just that, though: remember that in the repeated roots case, our complimentary solution has the form

$$y_c(t) = c_1e^{rt} + c_2te^{rt}.$$

If our $g(t) = e^{rt}$, multiplication by t won't be enough, since te^{rt} is also a term in the complimentary solution. So we'll actually need to multiply our guess by t^2, giving a final guess of

$$Y_p(t) = At^2e^{rt}.$$

Let's do some examples.

EXAMPLE 17.5. *Write down a guess for the form of a particular solution to the following differential equations.*

(1) $y'' - 3y' - 28y = 6t + e^{-4t} - 2$

First, we find the complimentary solution. It is

$$y_c(t) = c_1e^{7t} + c_2e^{-4t}.$$

Our nonhomogeneous term is $g(t) = 6t + e^{-4t} - 2$, which we can rearrange to group the polynomial terms together as $g(t) = 6t - 2 + e^{-4t}$. Thus our initial guess would be

$$At + B + Ce^{-4t}.$$

The first two terms aren't a problem, but the Ce^{-4t} term also appears in the complimentary solution. Since Cte^{-4t} doesn't show up in the complimentary solution, our final guess is

$$Y_p(t) = At + B + Cte^{-4t}.$$

(2) $y'' - 64y = t^2e^{8t} + \cos(t)$

The complimentary solution is

$$y_c(t) = c_1e^{8t} + c_2e^{-8t}.$$

Our initial guess for a particular solution is

$$(At + Bt + C)e^{8t} + D\cos(t) + E\sin(t).$$

If we distributed the exponential through the polynomial, we'd have a Ce^{8t} term which also showed up in our complimentary solution. What we'll need to do is to multiply the *entire* first term by t (to see why, just differentiate and plug in...you'll see that if we don't, we'll end up losing a coefficient which we'll need later). So our final guess is

$$Y_p(t) = t(At^2 + Bt + C)e^{8t} + D\cos(t) + E\sin(t).$$

(3) $y'' + 4y' = e^{-t}\cos(2t) + t\sin(2t)$

The complimentary solution is

$$y_c(t) = c_1\cos(2t) + c_2\sin(2t).$$

Our first guess for a particular solution would be

$$e^{-t}(A\cos(2t) + B\sin(2t)) + (Ct + D)\cos(2t) + (Et + F)\sin(2t).$$

First, we notice that despite having similar looking terms, we can't actually combine anything, since the similar terms are multiplied by factors which don't just differ by constant coefficients. Next, we notice that both the second and third terms involve components of the complimentary solution: $D\cos(2t)$ and $F\sin(2t)$. Thus we'll need to multiply those two terms by t. The first term is ok, though, since if we multiplied it out we would have a product of an exponential and a sine or a cosine, and those aren't terms in the complimentary solution. So we end up with

$$Y_p(t) = e^{-t}(A\cos(2t) + B\sin(2t)) + t(Ct + D)\cos(2t) + t(Et + F)\sin(2t).$$

(4) $y'' + 2 + 5 = e^{-t}\cos(2t) + t\sin(2t)$

Notice that the nonhomogeneous term in this example is the same as in the previous one; we've just changed the differential equation. Here, the complimentary solution is

$$y_c(t) = c_1e^{-t}\cos(2t) + c_2e^{-t}\sin(2t).$$

Our initial guess for the particular solution is the same as in the last example:

$$e^{-t}(A\cos(2t) + B\sin(2t) + (Ct + D)\cos(2t) + (Et + F)\sin(2t).$$

This time, the first term causes the problem, while the second and third are fine just as they are. So we'll multiply the first by t:

$$Y_p(t) = te^{-t}(A\cos(2t) + B\sin(2t)) + (Ct + D)\cos(2t) + (Et + F)\sin(2t).$$

(5) $y'' + 4y' + 4y = t^2e^{-2t} + 2e^{-2t}$

Here the complimentary solution is

$$y_c(t) = c_1e^{-2t} + c_2te^{-2t}.$$

Notice that we can factor out a e^{-2t} from our nonhomogeneous term, which then becomes $g(t) = (t^2 + 2)e^{-2t}$. This is the product of a polynomial and an exponential, so our initial guess is

$$(At^2 + Bt + C)e^{-2t}.$$

But this Ce^{-2t} term is the first term in our complimentary solution. Also, we have Bte^{-2t}, which is the second term in the complimentary solution. So this is no good. Next, we try multiplying by t:

$$t(At^2 + Bt + C)e^{-2t}.$$

This still causes problems: the Cte^{-2t} term is still the second term in our complimentary solution. If we multiply by t^2, though, we have no problems, and so our final guess is

$$Y_p(t) = t^2(At^2 + Bt + C)e^{-2t}.$$

$\square$