LECTURE 16

Undetermined Coefficients 2: Electric Boogaloo

1. Recap

Last class, we saw that if we have a nonhomogeneous equation

$$p(t)y'' + q(t)y' + r(t)y = g(t),$$

the general solution has the form

$$y(t) = y_c(t) + Y_p(t),$$

where $y_c(t) = c_1y_1(t) + c_2y_2(t)$, the *complimentary solution*, is the general solution to the homogeneous equation

$$p(t)y'' + q(t)y' + r(t)y = 0$$

and $Y_p(t)$, *a particular solution*, is some solution to the original equation.

When the coefficients of the homogeneous equation are constant, we can use the method of Undetermined Coefficients, which works for certain nonhomogeneous terms $g(t)$. With this method, we make a guess at the form of a particular solution $Y_p(t)$, leaving the coefficients undetermined. Then we plug this guess into the differential equation and do some algebra to calculate what the coefficients should be.

There are three basic types of nonhomogeneous terms that this method works for: exponentials, sines and cosines, and polynomials. If $g(t) = ae^{\alpha t}$, we make the guess $Y_p(t) = Ae^{\alpha t}$. If $g(t) = a\sin(\alpha t)$ or $a\cos(\alpha t)$, our guess is $Y_p(t) = A\cos(\alpha t) + B\sin(\alpha t)$.

2. Polynomials

The third and final basic class of nonhomogeneous term we can use this method with are polynomials.

Example 16.1. *Find a particular solution to*

$$y'' - 4y' - 12y = 3t^3 - 5t + 2.$$

In this case, $g(t)$ is a cubic polynomial. When we differentiate a polynomial, its order decreases. So if our initial guess is a general cubic, we should be able to capture all of the terms that will arise from differentiating. We guess

$$Y_p(t) = At^3 + Bt^2 + Ct + D.$$

Note that we have a t^2 term in our equation even though one doesn't appear in $g(t)$! Now, let's differentiate and plug in.

$$6At + 2B - 4\left(3At^2 + 2Bt + C\right) - 12\left(At^3 + Bt^2 + Ct + D\right) = 3t^3 - 5t + 2$$
$$-12At^3 + (-12A - 12B)\,t^2 + (6A - 8B - 12C)\,t + (2B - 4C - 12D) = 3t^3 - 5t + 2$$

We obtain a system of equations by setting coefficients equal.

$$\begin{array}{llcl} t^3: & -12A = 3 & \Rightarrow & A = -\frac{1}{4} \\ t^2: & -12A - 12B = 0 & \Rightarrow & B = \frac{1}{4} \\ t^1: & 6A - 8B - 12C = -5 & \Rightarrow & C = \frac{1}{8} \\ t^0: & 2B - 4C - 12D = 2 & \Rightarrow & D = -\frac{1}{6} \end{array}$$

So a particular solution is

$$Y_p(t) = -\frac{1}{4}t^3 + \frac{1}{4}t^2 + \frac{1}{8}t - \frac{1}{6}.$$

□

To be more general, if $g(t) = a_n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$, a polynomial of order n, then our guess is that a particular solution is also an nth degree polynomial, *i.e.*, $Y_p(t) = A_n t^n + A_{n-1}t^{n-1} + \ldots + A_1 t + A_0$.

Now, let's summarize what we've seen so far.

$\mathbf{g(t)}$	guess for $\mathbf{Y_p(t)}$
$ae^{\alpha t}$	$Ae^{\alpha t}$
$a\sin(\alpha t)$	$A\cos(\alpha t) + B\sin(\alpha t)$
$a\cos(\alpha t)$	$A\cos(\alpha t) + B\sin(\alpha t)$
$a_n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$	$A_n t^n + A_{n-1}t^{n-1} + \ldots + A_1 t + A_0$

We're now going to look at more complicated nonhomogeneous terms. For this method to work, they must always come from products and sums of the basic types. We'll start with products.

3. Products

The basic idea is that the guess for the product is the product of the guesses. We'll illustrate this through an example.

Example 16.2. *Find a particular solution to*

$$y'' - 4y' - 12y = te^{4t}.$$

Let's start by writing down the guesses for each of the individual pieces. In this example, $g(t)$ is the product of a 1st order polynomial, t, and an exponential, e^{4t}. The guess for the polynomial is $At + B$ while the guess for the exponential is Ce^{4t}. This yields a guess of

$$Ce^{4t}(At + B).$$

We want to minimize the number of constants, though, and we can see that

$$Ce^{4t}(At + B) = e^{4t}(ACt + BC).$$

This gives us really two constants, so we'll write the guess as

$$Y_p(t) = e^{4t}(At + B)$$

. Notice that this is just the guess for the t with an exponential tacked on.

Let's differentiate and plug in.

$$\begin{aligned} 16e^{4t}(At+B) + 8Ae^{4t} - 4\left(4e^{4t}(At+B) + Ae^{4t}\right) - 12e^{4t}(At+B) &= te^{4t} \\ (16A - 16A - 12A)\,te^{2t} + (16B + 8A - 16B - 4A - 12B)\,e^{4t} &= te^{4t} \\ -12Ate^{4t} + (4A - 12B)e^{4t} &= te^{4t} \end{aligned}$$

Then we set coefficients equal.

$$\begin{aligned} (te^{4t}): &\qquad -12A = 1 &\qquad \Rightarrow &\qquad A = -\frac{1}{12} \\ (e^{4t}): &\qquad (4A - 12B) = 0 &\qquad \Rightarrow &\qquad B = -\frac{1}{36} \end{aligned}$$

So, a particular solution for this differential equation is

$$Y_p(t) = e^{4t}\left(-\frac{1}{12}t - \frac{1}{36}\right) = -\frac{e^{4t}}{36}\left(3t+1\right).$$

□

The basic rule of thumb when we have a product involving an exponential is that we write down the guess for the other part of the product, then multiply that by the exponential without any leading coefficient. This allows us to not have to explicitly multiply through as we did in the previous example.

Let's do another.

EXAMPLE 16.3. *Find a particular solution to*

$$y'' - 4y' - 12y = 29e^{5t}\sin(3t).$$

Using the rule of thumb from before, we write down the guess for $\sin(3t)$ and tack on e^{5t}. This gives a guess of

$$Y_p(t) = e^{5t}(A\cos(3t) + B\sin(3t)).$$

So, let's differentiate and plug in.

$$\begin{aligned} &25e^{5t}(A\cos(3t) + B\sin(3t)) + 30e^{5t}(-A\sin(3t) + B\cos(3t)) + \\ &\qquad 9e^{5t}(-A\cos(3t) - B\sin(3t)) - 4(5e^{5t}(A\cos(3t) + B\sin(3t)) + \\ &\qquad\quad 3e^{5t}(-A\sin(3t) + B\cos(3t))) - 12e^{5t}(A\cos(3t) + B\sin(3t)) = 29e^{5t}\sin(3t) \end{aligned}$$

Next, gather like terms.

$$(-16A + 18B)e^{5t}\cos(3t) + (-18A - 16B)e^{5t}\sin(3t) = 29e^{5t}\sin(3t)$$

Set coefficients equal.

$$\begin{aligned} (e^{5t}\cos(3t)): &\qquad -16A + 18B = 0 \\ (e^{5t}\sin(3t)): &\qquad -18A - 16B = 29 \end{aligned}$$

This is solved by $A = -\frac{9}{10}$ and $B = -\frac{4}{5}$. So a particular solution to this differential equation is

$$Y_p(t) = e^{5t}\left(-\frac{9}{10}t - \frac{4}{5}\right) = -\frac{e^{5t}}{10}(9t+8).$$

□

Let's practice writing down the guesses for a couple of these.

EXAMPLE 16.4. *Write down the form of the particular solution to*

$$y'' - 4y' - 12y = g(t)$$

for the following $g(t)$s:

(1) $g(t) = \left(9t^2 - 103t\right)\cos(t)$

Here we've got the product of a quadratic and a cosine. The guess for the quadratic is

$$At^2 + Bt + C$$

and the guess for the cosine is

$$D\cos(t) + E\sin(t).$$

Multiplying the two guesses gives

$$\left(At^2 + Bt + C\right)(D\cos t) + \left(At^2 + Bt + C\right)(E\sin t)$$
$$\left(ADt^2 + BDt + CD\right)\cos t + \left(AEt^2 + BEt + CE\right)\sin t.$$

Each of our coefficients here is a product of two constants, which is just another constant. So, as before, to simplify everything, we'll replace each of those with a single constant to yield the following guess.

$$Y_p(t) = \left(At^2 + Bt + C\right)\cos t + \left(Dt^2 + Et + F\right)\sin t.$$

This is indicative of the general rule for a product of a polynomial and a trig function. Write down the guess for the polynomial, multiplied by a cosine, then add to that the guess for the polynomial (with different constants!) multiplied by a sine.

(2) $g(t) = e^{-2t}(3 - 5t)\cos(9t)$

This nonhomogeneous term has all three things. So, combining the two general rules from before, we get

$$Y_p(t) = e^{-2t}(At + B)\cos(9t) + e^{-2t}(Ct + D)\sin(9t).$$

□