LECTURE 15

Undetermined Coefficients

1. Nonhomogeneous Equations

We've finished discussing homogeneous equations. Now, let's turn to nonhomogeneous equations. A second order linear nonhomogeneous equation has the form

$$p(t)y'' + q(t)y' + r(t)y = g(t), \tag{15.1}$$

where $g(t) \neq 0$. How do we get the general solution to these?

Suppose we have two solutions $Y_1(t)$ and $Y_2(t)$. We should note that the Principle of Superposition no longer holds, as our equation is nonhomogeneous. As a result, we can't just take linear combinations of $Y_1(t)$ and $Y_2(t)$ and expect to get another solution. First, consider the equation

$$p(t)y'' + q(t)y' + r(t)y = 0, \tag{15.2}$$

which we will call the *associated homogeneous equation* to Equation 15.1. We then have the following result.

Theorem 15.1. *Suppose that $Y_1(t)$ and $Y_2(t)$ are two solutions to Equation 15.1 and that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to Equation 15.2. Then $Y_1(t) - Y_2(t)$ is a solution to Equation 15.2 and has the form*

$$Y_1(t) - Y_2(t) = c_1y_1(t) + c_2y_2(t).$$

Notice the notation used here. This will be standard for us: we will use capital letters to denote solutions to Equation 15.1 and lower case letters to denote solutions to Equation 15.2.

Now, how can we verify this theorem? Let's plug in $Y_1 - Y_2$ to Equation 15.2.

$$\begin{aligned} p(t)(Y_1 - Y_2)'' + q(t)(Y_1 - Y_2)' + r(t)(Y_1 - Y_2) &= 0 \\ (p(t)Y_1'' + q(t)Y_1' + r(t)Y_1) - (p(t)Y_2'' + q(t)Y_2' + r(t)Y_2) &= 0 \\ g(t) - g(t) &= 0 \qquad \text{since } Y_2 \text{ and } Y_2 \text{ solve (15.1)} \\ 0 &= 0 \end{aligned}$$

So, we have that $Y_1(t) - Y_2(t)$ solves Equation 15.2 Now, we know that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to Equation 15.2, and so any solution can be written as a linear combination of them. Thus, for some constants c_1 and c_2,

$$Y_1(t) - Y_2(t) = c_1y_1(t) + c_2y_2(t).$$

So the difference of any two solutions of Equation 15.1 is a solution to Equation 15.2. Let's suppose we've got one solution to Equation 15.1, which we'll denote by $Y_p(t)$. Let $Y(t)$ denote the general solution. We've just seen that we have

$$Y(t) - Y_p(t) = c_1y_1(t) + c_2y_2(t)$$

or

$$Y(t) = c_1y_1(t) + c_2y_2(t) + Y_p(t)$$

where y_1 and y_2 are a fundamental set of solutions to $Y(t)$. We will call

$$y_c(t) = c_1y_1(t) + c_2y_2(t)$$

the *complimentary solution* and $Y_p(t)$ a *particular solution*. So, the general solution can be expressed as

$$Y(t) = y_c(t) + Y_p(t).$$

Thus, to find the general solution to Equation 15.1, we'll need to find the general solution to Equation 15.2 (which we always know how to do, for example, in the constant coefficient case) and then find *some* solution to Equation 15.1. Adding these two pieces together will give us the general solution to Equation 15.1.

This should make sense. If we vary a solution to Equation 15.1 by just adding in some solution to Equation 15.2, it should still solve Equation 15.1, since the new term will contribute nothing when plugged into the equation. So there's complete freedom to change solutions by terms solving Equation 15.2, and so its general solution enters into the picture.

The outstanding question now is how to find some particular solution $Y_p(t)$ to Equation 15.1. There are two methods to do this. One of them, called Undetermined Coefficients, we will discuss at length: it reduces the problem entirely to an algebraic one, but it only works in very select circumstances. The other, called Variation of Parameters, we will touch on: it's a much more general method, but it requires integration which may not always be doable.

2. Undetermined Coefficients

Let's start with the method of Undetermined Coefficients. One disadvantage of this method is that it is only really useful for constant coefficient differential equations, so we'll restrict our attention to equations of the form

$$ay'' + by' + cy = g(t) \tag{15.3}$$

for $g(t) \neq 0$. The other main disadvantage of this method is that it only works for a fairly small group of $g(t)$s (though these functions include some of the most common ones).

Recall that what we are trying to do is to determine some particular solution $Y_p(t)$ to Equation 15.3 (there are in fact many different solutions to Equation 15.3: we're just after one). Then we can add that to the general solution to the associated homogeneous equation to get the general solution to Equation 15.3. The idea behind this method is that, for certain classes of nonhomogeneous terms, we're able to make a good educated guess as to how $Y_p(t)$ should look, up to some unknown coefficients (hence the name of the method). Then, we plug our guess into the differential equation and try to solve for the coefficients. If we can, our guess was correct and we have our $Y_p(t)$. If we can't solve for the coefficients, then we guessed incorrectly and we will need to try again.

3. The Basic Functions

There are three "groups" of basic types of nonhomogeneous terms $g(t)$ that are susceptible to this method: exponentials, trig functions (specifically, sin and cos), and polynomials. Once we see how these functions work, we'll be able to combine them in various ways.

3.1. Exponentials. Let's walk through an example where $g(t)$ is an exponential and see how the method works in this case.

Example 15.1. *Determine a particular solution to*

$$y'' - 4y' - 12y = 2e^{4t}.$$

How can we make a guess as to the form of $Y_p(t)$? When we plug $Y_p(t)$ into the equation, we should get $g(t) = 2e^{4t}$. We know that exponentials never appear or disappear during differentiation, so a reasonable guess might be

$$Y_p(t) = Ae^{4t}$$

for some coefficient A. Let's differentiate, plug in, and see if we can figure out what A must be.

If we plug into the differential equation, we get

$$\begin{aligned} 16Ae^{4t} - 4\left(4Ae^{4t}\right) - 12Ae^{4t} &= 2e^{4t} \\ -12Ae^{4t} &= 2e^{4t}. \end{aligned}$$

For these to be equal (and thus for our guess to be the solution), we'll need A to satisfy

$$-12A = 2 \qquad \Rightarrow \qquad A = -\frac{1}{6}.$$

So with this choice of A, our guess works, and the particular solution is

$$Y_p(t) = -\frac{1}{6}e^{4t}.$$

$\square$

Let's do an example of a full-blown initial value problem so that we can go through the whole process once. After that, we'll focus more on finding these particular solutions, since that's the main point of this method.

Example 15.2. *Solve the IVP*

$$y'' - 4y' - 12y = 2e^{4t} \qquad y(0) = -\frac{13}{6} \quad y'(0) = \frac{7}{3}.$$

We know that the general solution has the form

$$y(t) = y_c(t) + Y_p(t)$$

where the complimentary solution $y_c(t)$ is the general solution to the associated homogeneous equation

$$y'' - 4y' - 12y = 0$$

and $Y_p(t)$ is a particular solution to the original differential equation. From the previous example, we know that we can take

$$Y_p(t) = -\frac{1}{6}e^{4t}.$$

What is the complimentary solution? Our associated homogeneous equation has constant coefficients, so we need to find the roots of the characteristic equation.

$$\begin{aligned} r^2 - 4r - 12 &= 0 \\ (r-6)(r+2) &= 0. \end{aligned}$$

So we conclude that $r_1 = 6$ and $r_2 = -2$. These are distinct real roots, so the complimentary solution will be

$$y_c(t) = c_1e^{6t} + c_2e^{-2t}.$$

We must be careful here: the initial conditions are for the original, nonhomogeneous equation, *not for the associated homogeneous equation.* We absolutely do not want to apply them at this stage to y_c, since that is not the (or in fact, a) solution to the original equation.

Alright, so our general solution is the sum of $y_c(t)$ and $Y_p(t)$. We'll need it and its derivative to apply the initial conditions.

$$\begin{aligned} y(t) &= c_1e^{6t} + c_2e^{-2t} - \frac{1}{6}e^{4t} \\ y'(t) &= 6c_1e^{6t} - 2c_2e^{2t} - \frac{2}{3}e^{4t} \end{aligned}$$

Now we apply our initial conditions.

$$\begin{aligned} -\frac{13}{6} &= y(0) = c_1 + c_2 - \frac{1}{6} \\ \frac{7}{3} &= y'(0) = 6c_1 - 2c_2 - \frac{2}{3} \end{aligned}$$

This system is solved by $c_1 = -\frac{1}{8}$ and $c_2 = -\frac{15}{8}$, so our solution is

$$y(t) = -\frac{1}{8}e^{6t} - \frac{15}{8}e^{-2t} - \frac{1}{6}e^{4t}.$$

□

From now on, we'll focus more on just finding particular solutions, so don't forget how to solve IVPs!

3.2. Trig Functions. The second class of nonhomogeneous terms for which we can use this method are trig functions, specifically sin and cos.

EXAMPLE 15.3. *Find a particular solution for the following IVP:*

$$y'' - 4y' - 12y = 6\cos(4t).$$

In the first example, our nonhomogeneous term was an exponential, and we know that when we differentiate exponentials they persist. In this case, we've got a cosine function. When we differentiate a cosine, we get a sine. So we might expect our initial guess to require a sine term in addition to a cosine. Let's try it: set

$$Y_p(t) = A\cos(4t) + B\sin(4t).$$

Next, we differentiate and plug in.

$$\begin{aligned} -16A\cos(4t) - 16B\sin(4t) - 4\left(-4A\sin(4t) + 4B\cos(4t)\right) - 12\left(A\cos(4t) + B\sin(4t)\right) &= 13\cos(4t) \\ (-16A - 16B - 12A)\cos(4t) + (-16B + 16A - 12B)\sin(4t) &= 13\cos(4t) \\ (-28A - 16B)\cos(4t) + (16A - 28B)\sin(4t) &= 13\cos(4t) \end{aligned}$$

To solve for A and B, we now set coefficients equal. Note that the coefficient of $\sin(4t)$ on the right hand side is 0. So we get the system of equations

$$\begin{aligned} \cos(4t): &\qquad -28A - 16B = 13 \\ \sin(4t): &\qquad 16A - 28B \quad = 0. \end{aligned}$$

This system is solved by $A = -\frac{7}{20}$ and $B = -\frac{1}{5}$. So a particular solution is

$$Y_p(t) = -\frac{7}{20}\cos(4t) - \frac{1}{5}\sin(4t).$$

□

It's worth noting here that the guess would have been the same if $g(t)$ had been a sine instead of a cosine.

3.3. Polynomials. The third and final basic class of nonhomogeneous term we can use this method with are polynomials.

Example 15.4. *Find a particular solution to*

$$y'' - 4y' - 12y = 3t^3 - 5t + 2.$$

In this case, $g(t)$ is a cubic polynomial. When we differentiate a polynomial, its order decreases. So if our initial guess is a general cubic, we should be able to capture all of the terms that will arise from differentiating. We guess

$$Y_p(t) = At^3 + Bt^2 + Ct + D.$$

Note that we have a t^2 term in our equation even though one doesn't appear in $g(t)$! Now, let's differentiate and plug in.

$$\begin{aligned} 6At + 2B - 4\left(3At^2 + 2Bt + C\right) - 12\left(At^3 + Bt^2 + Ct + D\right) &= 3t^3 - 5t + 2 \\ -12At^3 + (-12A - 12B)\,t^2 + (6A - 8B - 12C)\,t + (2B - 4C - 12D) &= 3t^3 - 5t + 2 \end{aligned}$$

We obtain a system of equations by setting coefficients equal.

$$\begin{array}{lccc} t^3: & -12A = 3 & \Rightarrow & A = -\dfrac{1}{4} \\ t^2: & -12A - 12B = 0 & \Rightarrow & B = \dfrac{1}{4} \\ t^1: & 6A - 8B - 12C = -5 & \Rightarrow & C = \dfrac{1}{8} \\ t^0: & 2B - 4C - 12D = 2 & \Rightarrow & D = -\dfrac{1}{6} \end{array}$$

So a particular solution is

$$Y_p(t) = -\frac{1}{4}t^3 + \frac{1}{4}t^2 + \frac{1}{8}t - \frac{1}{6}.$$

□

4. Summary

Given each of the basic types, we make the following guesses.

$g(t)$	$Y_P(t)$ **guess**
$ae^{\alpha t}$	$Ae^{\alpha t}$
$a\cos(\alpha t)$	$A\cos(\alpha t) + B\sin(\alpha t)$
$a\sin(\alpha t)$	$A\cos(\alpha t) + B\sin(\alpha t)$
$a_n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$	$A_n t^n + A_{n-1}t^{n-1} + \ldots + A_1 t + A_0$

In future lectures we'll discuss how to handle sums and products of these basic types.