

LECTURE 30

Systems of Differential Equations

To this point, we've only discussed individual differential equations. But it's quite rare that a situation in the real world is modeled using only a single function: quite often, there are several interplaying factors at work in the evolution of something.

A good example is population dynamics. It would be possible to model the size of a single population using a single differential equation, making certain assumptions about death and birth rates (namely, that they are constant). But in general, this won't be the case: the death rate of a prey species is dependant on the size of a predator population and the size of the predator population will depend on the number of prey. To be able to write down a model for the size of the prey population, we need to know the predator population, and vice versa. This would then give us a system of two interlocked differential equations.

We'll turn our attention to this area next. An example of a system of first order linear equations is

$$\begin{aligned}x_1' &= x_1 + 4x_2 \\x_2' &= 2x_1 - 7x_2.\end{aligned}$$

We call a system like this *coupled* because we need to know what x_1 is to know what x_2 is and vice versa.

It's important to note that there will be a lot of similarities between our discussion here and our earlier discussion of second and higher order linear equations. There's a very good reason for this: any higher order linear equation can be written as a system of first order differential equations. Let's see how this is done.

EXAMPLE 30.1. Write the following second order differential equation as a system of first order linear differential equations.

$$3y'' + 2y' - 6y = 0 \quad y(2) = 0 \quad y'(2) = -2$$

All that's required to rewrite this equation as a first order system is a very simple change of variables. In fact, this is *always* the change of variables to use for a problem like this. We set

$$\begin{aligned}x_1(t) &= y(t) \\x_2(t) &= y'(t).\end{aligned}$$

Then we have

$$\begin{aligned}x_1' &= y' = x_2 \\x_2' &= y'' = 2y - \frac{2}{3}y' = 2x_1 - \frac{2}{3}x_2.\end{aligned}$$

Notice how we used the original differential equation to obtain the second equation. The first equation, $x_1' = x_2$, is always something you should expect to see when doing this, just by virtue of the change of variables we use.

All we have left to do is to convert the initial conditions.

$$\begin{aligned}x_1(2) &= y(2) = 0 \\x_2(2) &= y'(2) = -2\end{aligned}$$

Thus our original initial value problem has been transformed into the system

$$\begin{aligned}x_1' &= x_2 & x_1(2) &= 0 \\x_2' &= 2x_1 - \frac{2}{3}x_2 & x_2(2) &= -2\end{aligned}$$

□

Let's do an example to see how this works for higher order linear equations.

EXAMPLE 30.2. Write

$$y^{(4)} + 2y''' - y'' + 6y' - 3y = 0$$

as a system of first order differential equations.

We want to start by making an analogous change of variables as in Example 30.1. The only difference is that, since our equation in this example is fourth order, we will need four new variables instead of just two.

$$\begin{aligned}x_1 &= y \\x_2 &= y' \\x_3 &= y'' \\x_4 &= y'''\end{aligned}$$

Then we have

$$\begin{aligned}x_1' &= y' = x_2 \\x_2' &= y'' = x_3 \\x_3' &= y''' = x_4 \\x_4' &= y^{(4)} = 3y - 6y' + y'' - 2y''' = 3x_1 - 6x_2 + x_3 - 2x_4\end{aligned}$$

as our system of equations. □

To be able to solve these, we need to review some facts about systems of equations and linear algebra.

1. Systems of Equations

In this section, we will restrict our attention only to the linear algebra that might come up when studying systems of differential equations. This is far from a complete treatment, so if you're curious, taking a linear algebra course would be a good idea.

Suppose we start with a system of n equations with n unknowns, x_1, x_2, \dots, x_n .

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\&\vdots \\a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n\end{aligned}\tag{30.1}$$

Here's the basic fact about systems of equations with the same number of unknowns as equations, such as (30.1).

THEOREM 30.1. *Given a system of n equations with n unknowns, there are three possibilities for the number of solutions:*

- (1) *no solutions;*
- (2) *exactly one solution;*
- (3) *infinitely many solutions.*

We have one more definition to give: a system of equations such as (30.1) is called *nonhomogeneous* if at least one $b_i \neq 0$. If every $b_i = 0$, the system is called *homogeneous*. A homogeneous system has the following form.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= 0 \end{aligned} \tag{30.2}$$

Notice that there is always at least one solution, given by

$$x_1 = x_2 = \dots = x_n = 0.$$

This solution is called the *trivial solution*. This means that it is impossible for a homogeneous system to have zero solutions, and Theorem 30.1 can be modified as follows.

THEOREM 30.2. *Given a homogeneous system of n equations with n unknowns, there are two possibilities for the number of solutions:*

- (1) *exactly one solution, the trivial solution;*
- (2) *infinitely many non-zero solutions in addition to the trivial solution.*

2. Linear Algebra

While we could, in principle, solve the systems of equations (30.1) and (30.2) directly, we have some very powerful tools available to us. This is why linear algebra was invented. The main "objects" of study in linear algebra are *matrices* and *vectors*.

An n -times- n matrix (sometimes referred to as an n -dimensional matrix) is an array of numbers with n rows and n columns. It's possible to consider matrices with different numbers of rows and columns, but this is more general than we will need. An $n \times n$ matrix has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

There's one special matrix we will need to be familiar with; this is the n -dimensional *identity matrix*

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

We will be focusing on 2×2 matrices in this class; the principles of everything we will discuss extend to higher dimensional matrices, but the computations are much simpler in 2 dimensions.

Matrix addition and subtraction are fairly straightforward: everything is done componentwise. The same goes for multiplying a matrix by a constant, called *scalar* multiplication: we just multiply every component of the matrix by that constant. This will be illustrated in the following example.

EXAMPLE 30.3. Given the matrices

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -2 & 0 \\ 1 & 4 \end{pmatrix},$$

compute $A - 2B$.

The first thing to do is to compute $2B$.

$$2B = 2 \begin{pmatrix} -2 & 0 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 2 & 8 \end{pmatrix}$$

Then we have

$$\begin{aligned} A - 2B &= \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix} - \begin{pmatrix} -4 & 0 \\ 2 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 1 \\ -4 & -3 \end{pmatrix}. \end{aligned}$$

□

Notice that these operations require the dimensions of the matrices to be equal.

A *vector* is a one-dimensional array of numbers. For example,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

is a vector of n unknowns.

We can multiply two matrices A and B together by "multiplying" each row in A by each column in B .

EXAMPLE 30.4. Compute AB , where

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}.$$

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} (1)(0) + (2)(2) & (1)(1) + (2)(-3) \\ (-1)(0) + (3)(2) & (-1)(1) + (3)(-3) \end{pmatrix} \\ &= \begin{pmatrix} 4 & -5 \\ 6 & -10 \end{pmatrix} \end{aligned}$$

□

Notice that $AB \neq BA$. Matrix multiplication is *not* commutative. Also, the dimensions of the matrices being multiplied are important: if the number of columns of A do not match the number of rows of B , we cannot compute AB . Also, the identity matrix I_n is the identity for matrix multiplication; $I_n A = A I_n = A$ for any matrix A .

In particular, we can multiply an n -dimensional matrix and a vector with n components together as in the following example.

EXAMPLE 30.5. Compute

$$\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix}.$$

We proceed by "multiplying" each row in the matrix by the vector.

$$\begin{aligned} \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} &= \begin{pmatrix} (2)(-1) + (-1)(4) \\ 3(-1) + (2)(4) \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ 5 \end{pmatrix} \end{aligned}$$

□

REMARK. Multiplication of a matrix with a vector yields another vector. We then have an interpretation of a matrix A as a linear function on vectors (it's not hard to see that matrix multiplication breaks up over sums). This point of view is not essential, but is useful for what will follow over the next few lectures. It gets fleshed out more in a linear algebra course.

2.1. Determinants. Every square ($n \times n$) matrix has a number associated to it, called the *determinant*. We won't learn how to compute determinants for $n > 2$, as the process gets more and more complicated as n increases. The standard notation for the determinant of a matrix is

$$\det(A) = |A|.$$

For a 2×2 matrix, the determinant is computed using the following formula.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc; \quad (30.3)$$

that is, the determinant is the product of the main diagonal minus the product of the off diagonal.

EXAMPLE 30.6. Compute the determinants of

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

There's not much to do here but use (30.3).

$$\begin{aligned} \det(A) &= (2)(2) - (3)(1) = 4 - 3 = 1 \\ \det(B) &= (1)(4) - (2)(2) = 4 - 4 = 0 \end{aligned}$$

□

We call a matrix A *singular* if $\det(A) = 0$ and *nonsingular* otherwise. In the previous example, the first matrix was nonsingular while the second was singular.

Determinants give us important information about the existence of an inverse for a given matrix. The inverse of a matrix A , denoted A^{-1} , satisfies

$$AA^{-1} = A^{-1}A = I_n$$

. Inverses do not necessarily exist for a given matrix.

THEOREM 30.3. Given a matrix A ,

- (1) if A is nonsingular, an inverse A^{-1} will exist;
- (2) if A is singular, no inverse A^{-1} will exist.

3. Back to Systems

We return our attention now to the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \tag{30.4}$$

To express this system of equations in *matrix-vector form*, we start by writing both sides as vectors.

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Notice that the left hand side of the equation can be rewritten as a matrix-vector product.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

We can simplify this notation by writing

$$\mathbf{Ax} = \mathbf{b} \tag{30.5}$$

where \mathbf{x} is the vector whose entries are the unknowns in the system, A is the matrix of coefficients of the system (called the *coefficient matrix*), and \mathbf{b} is the vector whose entries are the right-hand sides of the equations. We call (30.5) the *matrix form* of the system of equations (30.4).

We know that the system of equations (30.4), and hence (30.5), have zero, one, or infinitely many solutions. We can actually go a little further. Suppose $\det(A) \neq 0$, *i.e.*, A is nonsingular. Then (30.5) has only one solution, namely,

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

So we can rewrite our earlier Theorem 30.1 in the following way.

THEOREM 30.4. *Given the system of equations (30.5),*

- (1) *if $\det(A) \neq 0$, there is exactly one solution;*
- (2) *if $\det(A) = 0$, there are either zero or infinitely many solutions.*

Recall that if our system (30.4) were homogeneous, *i.e.*, if each $b_i = 0$, we always have the trivial solution $x_i = 0$. Denoting the vector with entries all 0 by $\mathbf{0}$, the matrix form of a homogeneous system is

$$\mathbf{Ax} = \mathbf{0}. \tag{30.6}$$

Thus we can express the earlier Theorem 30.2 as follows.

THEOREM 30.5. *Given the homogeneous system of equations (30.6),*

- (1) *if $\det(A) \neq 0$, there is exactly one solution, $\mathbf{x} = \mathbf{0}$;*
- (2) *if $\det(A) = 0$, there will be infinitely many nonzero solutions.*