LECTURE 27

Inverse Transforms of Step Functions

Last class, we started to consider piecewise continuous functions. We saw that we could write them in terms of step functions $u_c(t)$ and that the Laplace transform

$$\mathcal{L}\left\{u_c(t)f(t-c)\right\} = e^{-cs}\mathcal{L}\left\{f(t)\right\},$$

where $f(t-c)$ is the coefficient function of $u_c(t)$ written as a function shifted by c.

Now, let's look at some inverse transforms. The previous formula's associated inverse transform is

$$\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\} = u_c(t)f(t-c), \tag{27.1}$$

where $f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}$. So here, we need to be careful about shifting: this time, though, we do it at the end, after finding the inverse transform of the coefficient of the exponential.

Example 27.1. *Find the inverse Laplace transform of the following.*

(i) $F(s) = \dfrac{se^{-4s}}{(2s-4)(s+3)}$

Whenever we do these, it's a good idea to ignore the exponential and determine the inverse transform of whatever's left over first. When it comes to splitting things up into terms, we should use our discretion: sometimes there's no way around it, but sometimes we can save ourselves work by not splitting things up. If you want, you can err on the side of treating everything separately. Then, by (27.1), the inverse transform of that term will be the appropriate step function multiplied by the *shifted* inverse transform we calculated.

In this case, we can't split anything up, as there's only one exponential and no terms without an exponential. So we'll pull out the exponential and ignore it for the time being:

$$F(s) = e^{-4s}\frac{s}{(2s-4)(s+3)} = e^{-4s}H(s).$$

We want to determine $h(t)$; once we have that, (27.1) tells us that the inverse transform will be

$$f(t) = h(t-4)u_4(t).$$

Now we need to partial fraction $H(s)$ so that we can take its inverse transform. The form of the decomposition is

$$H(s) = \frac{A}{2s-4} + \frac{B}{s+3} = \frac{A(s+3)+B(2s-4)}{(2s-4)(s+3)}.$$

Setting numerators equal, we have

$$s = A(s+3) + B(2s-4).$$

We can use the quick method of finding "key" values of s here.

$$\begin{aligned} s=2: &\qquad 2 = 5A &\qquad \Rightarrow &\qquad A = \frac{2}{5} \\ s=-3: &\qquad -3 = -10B &\qquad \Rightarrow &\qquad B = \frac{3}{10} \end{aligned}$$

So the partial fraction decomposition is

$$\begin{aligned} H(s) &= \frac{2}{5}\frac{1}{2s-4} + \frac{3}{10}\frac{1}{s+3} \\ &= \frac{2}{5}\frac{1}{2(s-2)} + \frac{3}{10}\frac{1}{s+3}. \end{aligned}$$

Notice here that we factored a 2 out of the denominator, so that we could directly take the inverse transform. Doing so, we have

$$h(t) = \frac{1}{5}e^{2t} + \frac{3}{10}e^{-3t}.$$

Let's return to the original problem. We wanted to find the inverse transform of

$$F(s) = e^{-4s}H(s).$$

This will be, by (27.1),

$$f(t) = h(t-4)u_4(t),$$

where $h(t)$ is what we just found above. So

$$\begin{aligned} f(t) &= u_4(t)\left(\frac{e^{2(t-4)}}{5} + \frac{3e^{-3(t-4)}}{10}\right) \\ &= u_4(t)\left(\frac{e^{2t-8}}{5} + \frac{3e^{-3t+12}}{10}\right). \end{aligned}$$

(ii) $G(s) = \dfrac{2e^{-3s} + e^{-7s}}{(s-3)(s^2+4)}$

As in the previous example, we want to begin by ignoring the exponentials. We could begin by writing

$$G(s) = e^{-3s}\frac{2}{(s-3)(s^2+4)} + e^{-7s}\frac{1}{(s-3)(s^2+4)},$$

then call the first fraction $F(s)$, the second $G(s)$, and begin doing partial fractions. But notice that if we pull out the constant from the first term, we have

$$G(s) = 2e^{-3s}\frac{1}{(s-3)(s^2+4)} + e^{-7s}\frac{1}{(s-3)(s^2+4)},$$

and the two fractions are in fact the same function. So we can save ourselves some effort by writing

$$G(s) = \left(2e^{-3s} + e^{-7s}\right)\frac{1}{(s-3)(s^2+4)} = \left(2e^{-3s} + e^{-7s}\right)H(s).$$

If you wanted to go the first route, that would be fine. It's best to use your discretion and go with whichever you're comfortable with.

Now we want to find the inverse transform of

$$H(s) = \frac{1}{(s-3)(s^2+4)}.$$

The partial fraction decomposition is

$$H(s) = \frac{A}{s-3} + \frac{Bs+C}{s^2+4} = \frac{A(s^2+4) + (Bs+C)(s-3)}{(s-3)(s^2+4)}.$$

Setting numerators equal and simplifying gives

$$\begin{aligned} 1 &= A(s^2+4) + (Bs+C)(s-3) \\ &= (A+B)s^2 + (-3B+C)s + (4A-3C). \end{aligned}$$

Setting coefficients equal and solving gives

$$\begin{aligned} s^2 &: \quad A + B = 0 \\ s^1 &: \quad -3B + C = 0 \\ s^0 &: \quad 4A - 3C = 1 \end{aligned} \qquad \Rightarrow \qquad A = \frac{1}{13} \qquad B = -\frac{1}{13} \qquad C = -\frac{3}{13}.$$

Substituting back into the transform (and pulling out the denominator of 13), we get

$$\begin{aligned} H(s) &= \frac{1}{13}\left(\frac{1}{s-3} + \frac{-s-3}{s^2+4}\right) \\ &= \frac{1}{13}\left(\frac{1}{s-3} - \frac{s}{s^2+4} - \frac{3}{2}\frac{2}{s^2+4}\right). \end{aligned}$$

Notice that, as in the previous lecture, we had to multiply the last term by $\frac{2}{2}$ so that we could get the correct denominator for inverse transforming. Now, if we take the inverse transform, we get

$$h(t) = \frac{1}{13}\left(e^{3t} - \cos(2t) - \frac{3}{2}\sin(2t)\right).$$

Returning to the original problem, we had

$$\begin{aligned} G(s) &= \left(2e^{-3s} + e^{-7s}\right) H(s) \\ &= 2e^{-3s}H(s) + e^{-7s}H(s). \end{aligned}$$

We had to distribute $H(s)$ through the parentheses to use (27.1), since we must end up with each term containing one step function and one coefficient function. By (27.1), we have

$$\begin{aligned} g(t) &= 2h(t-3)u_3(t) + h(t-7)u_7(t) \\ &= \frac{1}{13}\left(2e^{3t-9} - 2\cos(2t-6) - 3\sin(2t-6) + e^{3t-21} - \cos(2t-14) - \frac{3}{2}\sin(2t-14)\right). \end{aligned}$$

(iii) $F(s) = \dfrac{4s + 2e^{-4s}}{s^2(s-2)}$

Here we will first have to break up our transform into two pieces, since one term has a constant and the other has an s. The exponential is, for these considerations, irrelevant. We write

$$F(s) = \frac{4s}{s^2(s-2)} + e^{-4s}\frac{2}{s^2(s-2)} = F_1(s) + e^{-4s}F_2(s).$$

We'll need to partial fraction the functions $F_1(s)$ and $F_2(s)$ separately. Let's consider $F_1(s)$ first.

$$F_1(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2}$$

$$\begin{aligned} 4s &= As(s-2) + B(s-2) + Cs^2 \\ &= (A+C)s^2 + (-2A+B)s - 2B \end{aligned}$$

Now we find the constants

$$\begin{aligned} A + C &= 0 \\ -2A + B &= 4 \\ -2B &= 0 \end{aligned} \qquad \Rightarrow \qquad A = -2 \quad B = 0 \quad C = 2$$

So $F_1(s)$ and its inverse transform are

$$\begin{aligned} F_1(s) &= -\frac{2}{s} + \frac{2}{s-2} \\ f_1(t) &= -2 + 2e^{2t} \end{aligned}$$

Now we'll repeat the process for $F_2(s)$.

$$F_2(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2}$$

$$\begin{aligned} 2 &= As(s-2) + B(s-2) + Cs^2 \\ &= (A+C)s^2 + (-2A+B)s - 2B \end{aligned}$$

So we have

$$\begin{aligned} A + C &= 0 \\ -2A + B &= 0 \\ -2B &= 2 \end{aligned} \qquad \Rightarrow \qquad A = -\frac{1}{2} \quad B = -1 \quad C = \frac{1}{2}.$$

Thus $F_2(s)$ and its inverse transform are

$$\begin{aligned} F_2(s) &= -\frac{1}{2}\frac{1}{s} - \frac{1}{s^2} + \frac{1}{2}\frac{1}{s-2} \\ f_2(t) &= -\frac{1}{2} - t + \frac{1}{2}e^{2t}. \end{aligned}$$

Our original transformed function was

$$F(s) = F_1(s) + e^{-4s}F_2(s).$$

Now, by (27.1), our inverse transform will be

$$\begin{aligned} f(t) &= f_1(t) + f_2(t-4)u_4(t) \\ &= -2 + 2e^{2t} + \left(-\frac{1}{2} - (t-4) + \frac{1}{2}e^{2(t-4)}\right)u_4(t) \\ &= -2 + 2e^{2t} + \left(\frac{7}{2} - t + \frac{1}{2}e^{2t-8}\right)u_4(t) \end{aligned}$$

(iv) $G(s) = \dfrac{2 - se^{-2s}}{s^2 - 2s + 10}$

In this case, we won't have to do partial fractions, since the denominator doesn't factor. Instead, we'll have to complete the square, and we get

$$G(s) = \frac{2 - se^{-2s}}{(s-1)^2 + 9} = \frac{2}{(s-1)^2 + 9} - e^{-2s}\frac{s}{(s-1)^2 + 9} = G_1(s) - e^{-2s}G_2(s).$$

As in the last example, we need to treat $G_1(s)$ and $G_2(s)$ separately. For $G_1(s)$, we need the numerator to involve 3, so we'll multiply top and bottom by $\frac{3}{3}$.

$$\begin{aligned} G_1(s) &= \frac{2}{(s-1)^2 + 9} \\ &= \frac{2}{3}\frac{3}{(s-1)^2 + 9} \\ g_1(t) &= \frac{2}{3}e^t \sin(3t) \end{aligned}$$

For $G_2(s)$, however, we need to use a trick from last class. We need the numerator to involve $s-1$, but there is only an s. So we have to add and subtract 1.

$$\begin{aligned}
G_2(s) &= \frac{s}{(s-1)^2+9} \\
&= \frac{s-1+1}{(s-1)^2+9} \\
&= \frac{s-1}{(s-1)^2+9} + \frac{1}{(s-1)^2+9} \\
&= \frac{s-1}{(s-1)^2+9} + \frac{1}{3}\frac{3}{(s-1)^2+9} \\
g_2(t) &= e^t\cos(3t) + \frac{1}{3}e^t\sin(3t)
\end{aligned}$$

Our original transform was

$$G(s) = G_1(s) + e^{-2s}G_2(s).$$

By (27.1),

$$\begin{aligned}
g(t) &= g_1(t) + g_2(t-2)u_2(t) \\
&= \frac{2}{3}e^t\sin(3t) + \left(e^{t-2}\cos(3(t-2)) + \frac{1}{3}e^{t-2}\sin(3(t-2))\right)u_2(t) \\
&= \frac{2}{3}e^t\sin(3t) + \left(e^{t-2}\cos(3t-6) + \frac{1}{3}e^{t-2}\sin(3t-6)\right)u_2(t).
\end{aligned}$$

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