

LECTURE 26

Step Functions

When we defined the Laplace transform, we made mention of the fact that we didn't have to transform only continuous functions; we could also transform functions that were *piecewise* continuous. So far, however, all of the examples of transforms and inverse transforms we've considered have involved only continuous functions.

It would be nice to be able to solve differential equations with forcing functions that weren't continuous, but had isolated points of discontinuity where the forcing function jumped from one value to another abruptly. We can easily imagine applications where these would arise: a mechanical vibration where we add some extra force later on or an electrical circuit where a voltage is switched on or off at some specified time.

Without Laplace transforms, we would have to split these initial value problems into several different problems, with each subsequent initial condition derived from the earlier solutions. But with Laplace transforms, we can deal with them in a single stroke.

1. Step Functions

Consider the following function.

$$u(t - c) = u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

This function is called the *step* or *Heaviside function* at c . It represents a jump at $t = c$ from zero (which means anything multiplied by it doesn't contribute anything prior to time c) to one at $t = c$ (where any coefficient it has will begin to contribute). You can think of a step function as a switch that turns on its coefficient at a specified time.

The step function itself takes only values of 0 or 1, but it's easy enough to make it give any value we desire. For example, $4u_c(t)$ will contribute 4 once $t = c$ and beyond while contributing nothing earlier, while $-7u_c(t)$ will contribute -7 once t hits c and nothing earlier.

We can produce a switch that turns off at $t = c$. If we consider the function

$$1 - u_c(t) = \begin{cases} 1 - 0 = 1 & \text{if } t < c \\ 1 - 1 = 0 & \text{if } t \geq c \end{cases},$$

it will exhibit precisely this sort of behavior. We use this, again, to get any contribute we might want only *prior* to $t = c$: e.g., $3 - 3u_c(t)$ will give us a value of 3 when $t < c$ and 0 when $t \geq c$.

Step functions are the key to neatly writing down piecewise continuous functions as a single expression, rather than as a system of cases.

Example 26.1. Write the following function in terms of step functions.

$$f(t) = \begin{cases} 9 & \text{if } t < 2 \\ -6 & \text{if } 2 \leq t < 6 \\ 25 & \text{if } 6 \leq t < 9 \\ 7 & \text{if } 10 \leq t \end{cases}$$

There are three jumps in this function: at $t = 2$, $t = 6$, and $t = 9$. So we will need a total of three step functions, each of which will correspond to one of these jumps. In terms of step functions,

$$f(t) = 9 - 15u(t - 2) + 31u(t - 6) - 18u(t - 9).$$

How did we find that?

When $t < 2$, all of the step functions have a value of 0. So the only contributing term in our expression for $f(t)$ is 9, and on this region $f(t) = 9$.

On the next interval, $2 \leq t < 6$, we want $f(t) = -6$. The first step function $u(t - 2)$ is on, while the others are off. Notice that the 9 term is still contributing, as nothing gets turned off. As a result, the coefficient of $u(t - 2)$ will need to cause the sum of it and 9 to equal -6. Thus it must be -15.

On the third interval, $6 \leq t < 9$, we now have two “on” step functions while the last one is off. The first two terms, the 9 and the $-15u(t - 2)$, continue to contribute, as again they never turn “off.” Thus the coefficient of the newly contributing step function, $u(t - 6)$, will need to combine with these to give us our desired value $f(t) = 25$. Thus it must be 31.

Lastly, we have the interval $9 < t$. Now all of the terms contribute, and the coefficient of $u(t - 9)$, the step function corresponding to the final jump, should move us from our previous value of 25 to our new value, $f(t) = 7$. As a result, it must be -18 .

So the function written above has the right value on all intervals. \square

We’re not usually interested just in situations where our forcing function takes constant values on intervals, however. In the case of mechanical vibrations of the sort we considered earlier, we might want to add in a new external force which is sinusoidal. Or a nonconstant voltage might be switched on at some time after we’ve started considering an electrical circuit.

So we will consider the following piecewise continuous function: $g(t) = u(t - c)f(t - c)$, where $f(t)$ is some function. We shift it by c , the starting point of the step function, to indicate that we want it to start working at $t = c$ instead of $t = 0$, which it would normally. Think of this graphically: to get the graph of $g(t)$, what we want to do is take the graph of $f(t)$, starting at $t = 0$, and “push” it to start at $t = c$ with a value of 0 prior to this time. This requires us to shift the argument of f by c .

2. Laplace Transform

What is the Laplace transform $\mathcal{L}\{g(t)\}$?

$$\begin{aligned} \mathcal{L}\{g(t)\} &= \mathcal{L}\{u(t - c)f(t - c)\} \\ &= \int_0^{\infty} u(t - c)e^{-st}f(t - c) dt \\ &= \int_c^{\infty} e^{-st}f(t - c) \quad \text{using the definition of the step function.} \end{aligned}$$

Now, this looks sort of like a Laplace transform, except that the integral starts at $t = c$ instead of $t = 0$. So we'll introduce a new variable $u = t - c$ to shift the integral to start at 0.

$$\begin{aligned} G(s) &= \int_0^\infty e^{-s(u+c)} f(u) du \\ &= e^{-sc} \int_0^\infty e^{-su} f(u) du \\ &= e^{-sc} F(s) \quad \text{using the notation } F(s) = \mathcal{L}\{f(u)\}. \end{aligned}$$

Notice that the Laplace transform we end up with is the Laplace transform of the *original* function $f(t)$ multiplied by an exponential related to the step function's "on time", even though we had shifted the function by c to begin with. Summarizing, we have the formula

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-sc}F(s) = e^{-sc}\mathcal{L}\{f(t)\}. \quad (26.1)$$

It is *critical* that we write the function to be transformed in the correct form, as a different function shifted by c , before we transform by using (26.1). Again, we also cannot note strongly enough that when using this formula, we end up computing the transform of $f(t)$, *not* the shifted function $f(t-c)$. This is something that can be a sticking point initially, but with practice it usually becomes clear.

We can, of course, using (26.1) to get a formula for a step function by itself. To do so, we consider a step function multiplied by the constant function $f(t) = 1$. In this case, $f(t-c) = 1$ as well, since it doesn't matter what value of t we input into f , we still get the same output 1. Doing this gives

$$\mathcal{L}\{u_c(t)\} = \mathcal{L}\{u_c(t) \cdot 1\} = e^{-cs}\mathcal{L}\{1\} = \frac{1}{s}e^{-cs}. \quad (26.2)$$

Example 26.2. Find the Laplace transforms of each of the following.

(i) $f(t) = 10u_6(t) + 3(t-4)^2u_4(t) - (1 + e^{10-2t})u_5(t)$

Recall that we must write each piece in the form $u_c(t)h(t-c)$ before we take the transform. If it isn't in that form, we have to put it in that form.

There are three terms in $f(t)$. We will use the linearity of the Laplace transform to treat them separately, then add them all together in the end. Let's write

$$f(t) = f_1(t) + f_2(t) + f_3(t)$$

to keep ourselves organized.

$f_1(t) = 10u_6(t)$, so it is just a constant times a step function. We can thus use (26.2) to determine its Laplace transform.

$$\mathcal{L}\{f_1(t)\} = 10\mathcal{L}\{u_6(t)\} = \frac{10e^{-6s}}{s}$$

$f_2(t) = 3(t-4)^2u_4(t)$, so we have to do two things: write it as a function shifted by 4 (if it isn't in that form already) and isolate the function that was shifted and transform it. In this case, we're good: we can write $f_2(t) = h(t-4)u_4(t)$, with $h(t) = 3t^2$. Thus

$$\mathcal{L}\{f_2(t)\} = e^{-4s}\mathcal{L}\{3t^2\} = 3e^{-4s}\frac{2}{s^3} = \frac{6e^{-4s}}{s^3}.$$

Finally, we have $f_3(t) = -(1 + e^{10-2t})u_5(t)$. Again, we have to express it as a function shifted by 5 and then identify the unshifted function so that we may transform it. This can

be accomplished by rewriting

$$f_3(t) = - \left(1 + e^{-2(t-5)} \right) u_5(t),$$

so, writing $g_3(t) = h(t-5)u_5(t)$, we have $h(t) = -(1 + e^{-2t})$ as the unshifted coefficient function. Thus

$$\mathcal{L}\{f_3(t)\} = e^{-5s} \mathcal{L}\{-(1 + e^{-2t})\} = -e^{-5s} \left(\frac{1}{s} + \frac{1}{s+2} \right).$$

Putting it all together,

$$F(s) = \frac{10e^{-6s}}{s} + \frac{6e^{-4s}}{s^3} - e^{-5s} \left(\frac{1}{s} + \frac{1}{s+2} \right).$$

(ii) $g(t) = t^2 u_2(t) - \cos(t) u_7(t)$.

In the last example, it turned out that all of the coefficient functions were pre-shifted (the most we had to do was pull out a constant to see that). In this example, that is definitely not the case. So what we want to do is to write each of our coefficient functions as the shift (by whichever constant is appropriate for that step function) of a *different* function. The idea is that we add and subtract the desired quantity, then simplify, keeping the correct shifted term.

So, let's write $g(t) = g_1(t) + g_2(t)$. $g_1(t) = t^2 u_2(t)$.

$$g_1(t) = (t - 2 + 2)^2 u_2(t)$$

This isn't quite right, but we'll use the associativity property of addition:

$$\begin{aligned} g_1(t) &= ((t - 2) + 2)^2 u_2(t) \\ &= (t - 2)^2 + 4(t - 2) + 4 u_2(t). \end{aligned}$$

Now we can see that $g_1(t) = h(t-2)u_2(t)$, where $h(t) = t^2 + 4t + 4$.

$$\mathcal{L}\{g_1(t)\} = e^{-2s} \mathcal{L}\{t^2 + 4t + 4\} = e^{-2s} \left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right)$$

The second term is similar. We start with

$$g_2(t) = -\cos(t)u_7(t) = -(\cos((t-7)+7)u_7(t))u_7(t).$$

Here we need to use the trig identity

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

This yields

$$g_2(t) = -(\cos(t-7)\cos(7) - \sin(t-7)\sin(7))u_7(t).$$

Since $\cos(7)$ and $\sin(7)$ are just constants (and ones we won't bother evaluating, since they're not nice values), we get (after using the linearity of the Laplace transform)

$$\begin{aligned} \mathcal{L}\{g_2(t)\} &= -e^{-7s} (\cos(7)\mathcal{L}\{\cos(t)\} - \sin(7)\mathcal{L}\{\sin(t)\}) \\ &= -e^{-7s} \left(\frac{s \cos(7)}{s^2 + 1} - \frac{\sin(7)}{s^2 + 1} \right). \end{aligned}$$

Piecing everything back together, we get

$$G(s) = e^{-2s} \left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right) - e^{-7s} \left(\frac{s \cos(7) - \sin(7)}{s^2 + 1} \right).$$

$$(iii) f(t) = \begin{cases} t^3 & \text{if } t < 4 \\ t^3 + 2 \sin\left(\frac{t}{12} - \frac{1}{3}\right) & 4 \leq t \end{cases}$$

The first step here is to write $f(t)$ as a single expression using step functions.

$$f(t) = t^3 + 2 \sin\left(\frac{t}{12} - \frac{1}{3}\right) u_4(t)$$

Next, we want to write the coefficients of $u_4(t)$ as another function shifted by 4.

$$f(t) = t^3 + 2 \sin\left(\frac{1}{12}(t - 4)\right) u_4(t).$$

Since everything is appropriately shifted, we have

$$\begin{aligned} F(s) &= \mathcal{L}\{t^3\} + 2e^{-4s} \mathcal{L}\left\{\sin\left(\frac{1}{12}t\right)\right\} \\ &= \frac{3!}{s^4} + 2e^{-4s} \frac{\frac{1}{12}}{s^2 + \left(\frac{1}{12}\right)^2} \\ &= \frac{6}{s^4} + \frac{e^{-4s}}{12\left(s^2 + \frac{1}{144}\right)} \\ &= \frac{6}{s^4} + \frac{e^{-4s}}{12s^2 + \frac{1}{12}}. \end{aligned}$$

$$(iv) g(t) = \begin{cases} t & \text{if } t < 2 \\ 3 + (t - 2)^2 & \text{if } 2 \leq t \end{cases}$$

First, we need to write $g(t)$ using step functions.

$$g(t) = t + (-t + 3 + (t - 2)^2) u_2(t).$$

Notice that we had to subtract t from the coefficient of $u_2(t)$ in order to make $g(t)$ have the correct value when $t \geq 2$. However, this means that the coefficient function of $u_2(t)$ is no longer properly shifted. As a result, we need to add and subtract 2 from that t to make it have the proper form.

$$\begin{aligned} g(t) &= t + (-t - 2 + 2) + 3 + (t - 2)^2 u_2(t) \\ &= t + (-t - 2) - 2 + 3 + (t - 2)^2 u_2(t) \\ &= t + (-t - 2) + 1 + (t - 2)^2 u_2(t) \end{aligned}$$

So, we have

$$\begin{aligned} G(s) &= \mathcal{L}\{t\} + e^{-2s} (\mathcal{L}\{-t\} + \mathcal{L}\{1\} + \mathcal{L}\{t^2\}) \\ &= \frac{1}{s^2} + e^{-2s} \left(-\frac{1}{s^2} + \frac{1}{s} + \frac{2}{s^3}\right). \end{aligned}$$

□

As you can see, taking Laplace transforms of functions involving step functions can be a bit more complicated than taking Laplace transforms of other functions. It's still not too bad, we just have to be careful to make sure that our coefficient functions are appropriately shifted.